Let's consider the cllxn of all fxns $G \to \mathbb{C}$. Since G is finite, this cllxn is a fin-dim vecspace ($\dim = |G|$). In fact,

$$\{f : G \to \mathbb{C}\} \cong \mathbb{C}G$$
$$f \longmapsto \sum_{g \in G} f(g)\, g.$$

$\leftarrow$ as vecspaces. Different ring structures!

Prop'n Define

$$\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

This is a Hermitian inner product on $\mathbb{C}G$. Moreover,

$$\mathbb{C}G \otimes \mathbb{C}G \xrightarrow{\langle , \rangle} \mathbb{C}$$

is G-equivariant (w/ trivial action on $\mathbb{C}$).

Pf of Prop'n:

- $\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum \overline{f_1(g) \overline{f_2(g)}}$

$= \overline{\frac{1}{|G|} \sum \overline{f_1(g)} f_2(g)}$

$= \overline{\langle f_2, f_1 \rangle}.$

- $\langle f, f \rangle = \frac{1}{|G|} \sum \|f(g)\|^2$

> 0 if $f \neq 0$

- $\langle hf_1, hf_2 \rangle = \frac{1}{|G|} \sum_g f_1(h^{-1}g) \cdot \overline{f_2(h^{-1}g)}$

$= \frac{1}{|G|} \sum_x f_1(x) \overline{f_2(x)}$

$= \langle f_1, f_2 \rangle.$ //

Rmk Under the isomorphism

$$\{f : G \to \mathbb{C}\} \cong \mathbb{C}G$$

the induced (left) G-action on $\{f\}$ is

$$hf : g \longmapsto f(h^{-1}g),$$

since

$$h\left(\sum f(g)\, g\right) = \sum f(g)\, hg$$
$$= \sum_x f(h^{-1}x)\, x.$$

Sanity check:

$$((h_1 h_2) f)(g) = f\left((h_1 h_2)^{-1} g\right)$$
$$= f(h_2^{-1} h_1^{-1} g)$$
$$= (h_2 f)(h_1^{-1} g)$$
$$= (h_1(h_2 f))(g).$$

Remember: We're trying to prove

$$\#\{\text{irreps}\}/_{\text{iso}} = \# \frac{G}{G}.$$

We don't yet even know that the lefthand side is finite; until now.

Prop. U, V irrep

$$\Longrightarrow \langle \chi_U, \chi_V \rangle = \begin{cases} 1 & U \cong V \\ 0 & \text{other.} \end{cases}$$

Pf

$$\langle \chi_U, \chi_V \rangle = \frac{1}{|G|} \sum_g \chi_U(g) \overline{\chi_V(g)}$$

$$= \frac{1}{|G|} \sum_g \chi_U(g) \chi_{V^\vee}(g)$$

$$= \frac{1}{|G|} \sum_g \chi_{U \otimes V^\vee}(g)$$

$$= \frac{1}{|G|} \sum_g \chi_{\hom_{\mathbb{K}}(V,U)}(g)$$

Lemma $= \dim\left(\hom_{\mathbb{K}}(U,V)^G\right)$

HW $= \dim\left(\hom_G(U,V)\right)$

Schur's Lemma $= \begin{cases} 1 & U \cong V \\ 0 & \text{other} \end{cases}$ //

Lemma W G-rep, $\chi_W : G \longrightarrow \mathbb{C}$ associated character. Then

$$\frac{1}{|G|} \sum_g \chi_W(g) = \dim W^G.$$

$W^G = \{w \mid gw = w\}$

Pf Consider map

$$W \xrightarrow{\pi} W^G$$

$$w \longmapsto \frac{1}{|G|} \sum_{g \in G} gw.$$

Since $\pi^2 = \pi$, $\pi = \begin{pmatrix} I & \\ & 0 \end{pmatrix}$ in some basis, hence

$$tr(\pi) = \dim W^G.$$

OTOH, $\pi = \frac{1}{|G|} \sum_g g \in End(W)$

hence

$$tr(\pi) = \frac{1}{|G|} \sum_g tr(g)$$

$$= \frac{1}{|G|} \sum_g \chi_W(g). //$$

This admits a slew of useful facts:

<u>Cor</u> The decomposition

$$W = \bigoplus W_\alpha$$

of any Grep into irreps is unique, in that the # of times a given irrep appears in the $\bigoplus$ is fixed.

<u>Pf</u> $W \cong \bigoplus W_\alpha^{n_\alpha} \Rightarrow \chi_W = \sum n_\alpha \chi_\alpha$

$\Rightarrow \langle \chi_W, \chi_{W_\alpha} \rangle = n_\alpha$

$=$ # of times an irrep V appears in $\{W_\alpha\}$ //

<u>Cor</u> Every irrep W_α appears in the regular rep $\dim W_\alpha$ times.

<u>Pf</u> $\langle \chi_{W_\alpha}, \chi_{reg} \rangle = \frac{1}{|G|} \sum_g \chi_{W_\alpha}(g) \chi_{reg}(g)$

$$= \frac{1}{|G|} \chi_{W_\alpha}(e) \chi_{reg}(e) + \frac{1}{|G|} \sum_{g \neq e} \chi_{W_\alpha}(g) \chi_{reg}(g)$$

$$= \frac{1}{|G|} \dim W_\alpha \cdot \dim \chi_{reg} + \frac{1}{|G|} \sum_{g \neq e} \chi_{W_\alpha}(g) \cdot 0$$

$$= \dim W_\alpha. \text{ //}$$

Hence the cllxn

$\{\chi_U\}_{U \text{ isom class of irrep}}$

is an orthonormal collection of vectors in $\mathbb{C}G \cong \mathbb{C}^{|G|}$, and we conclude $\#\{\text{irreps}\}/_{\cong} \leq |G|$.

We can do even better:

<u>Defn</u> A fxn $\alpha: G \to \mathbb{C}$ is called a <u>class</u> <u>fnction</u> if

$$\alpha(ghg^{-1}) = \alpha(h) \quad \forall h, g \in G.$$

Note the cllxn of class fxns forms a subspace of all fnctions. Restricting $\langle , \rangle$ to the class fxns, and noting $\dim\{\alpha \mid \alpha \text{ class fxn}\}$

$$= \dim\left(\mathbb{C}^{\frac{G}{G}}\right)$$

$$= \# \frac{G}{G},$$

we have:

<u>Cor</u> $\#\{\text{irreps}\}/_{\text{isom}} \leq \# \frac{G}{G}$.

Now we need to show equality.

Lemma Let $f: G \to \mathbb{C}$ be a class fxn. Then for any G-rep V,

$$\sum_{g \in G} f(g)\, g : V \to V$$

is a map of G-reps.

Conversely, if a fxn f defines a map of G-reps via the above formula for every G-rep V, f is a class fxn.

Pf Let $f^G := \sum f(g)\, g \in \mathrm{End}(V)$.

Then
$$\begin{aligned} h f^G(v) &= \sum h f(g)\, g(v) \\ &= \sum f(g) \cdot hg\,(v) \\ &= \sum f(g) \cdot hgh^{-1} \cdot hv \\ &= \sum_x f(h^{-1}xh) \cdot x \cdot hv \\ &= \sum f(x)\, x\,(hv) \\ &= f^G(hv). \end{aligned}$$

Conversely, let V be the std rep.

Then $h f^G(e_x) = f^G(he_x) = f^G(e_{hx})$

means
$$h \sum f(g)\, g e_x = \sum f(g)\, g\, e_{hx}$$

ie.
$$\sum f(g)\, e_{hgx} = \sum f(g)\, e_{ghx}$$

Recall $\{e_g\}_{g \in G}$ form a basis!

Setting $x = h^{-1}$, conclude
$$f(h^{-1}gh) = f(g). \quad //$$

Rmk Let R be a ring, not necessarily commutative. Recall a left R-module is an abelian group M equipped w/ a map

$$R \times M \longrightarrow M$$
$$(r, x) \longmapsto rx$$

s.t.

$$(rs)x = r(sx)$$
$$(r+s)x = rx + sx$$
$$r(x+y) = rx + ry$$
$$1x = x.$$

$\iff$ $r: M \longrightarrow M$, $x \longmapsto rx$ defines ring homom $R \longrightarrow \mathrm{End}(M)$

A map of R-modules is a group homomorphism

$$\phi: M \to N$$

that respects the module action:

$$\phi(rx) = r\phi(x).$$

Q: For which $r \in R$ is the scaling map

$$\phi_r: M \longrightarrow M$$
$$x \longmapsto rx$$

a module homomorphism $\forall M$?

A: $\phi_r(sx) = r(sx) = (rs)x$

$s\phi_r(x) = s(rx) = (sr)x.$

Taking $M = R$ w/ multiplication as module action, we see ϕ_r is a module map iff r is in the center of R.

Cor The center of $\mathbb{C}G$ is exactly the class functions. In particular, if ϕ, ψ are class fxns, so is $\phi \cdot \psi$.

Multiplication in $\mathbb{C}G$ via gp ring structure.

Cor Let Φ be a class fxn. If

$$\langle \Phi, \chi_V \rangle = 0$$

$\forall$ irreps V, then $\Phi = 0$.

Pf $\Phi_V : V \longrightarrow V$ is a map of G-representations, given by

$$v \longmapsto \sum_g \Phi(g) \cdot gv.$$

Since V is irreducible, by Schur's Lemma,

$$\sum_g \Phi(g) \cdot gv = \lambda v \qquad \text{HW 1}$$

for some λ independent of $v \in V$. Thus

$$\mathrm{tr}(\Phi_V) = \lambda \cdot \dim V$$

On the other hand,

$$\begin{aligned}
\langle \Phi, \chi_V \rangle &= \frac{1}{|G|} \sum_{g \in G} \Phi(g) \chi_V(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \Phi(g)\, \mathrm{trace}(\rho(g)) \\
&= \frac{1}{|G|} \mathrm{trace}\Big(\sum_{g \in G} \Phi(g) \rho(g) \Big) \\
&= \frac{\lambda \cdot \dim V}{|G|}.
\end{aligned}$$

The LHS is zero, so $\lambda = 0$.

Setting W = regular rep., note that

$$\Phi_W(e_h) = \sum \Phi(g)\, e_{gh}.$$

So Φ_W is the zero map iff $\Phi(g) = 0 \ \forall g$.

But if Φ_W has non-trivial image, project it to some irreducible component:

$$\begin{array}{ccc}
W & \xrightarrow{\Phi_W} & W \\
 & & \downarrow \\
 & & W_\alpha \\
 & & \text{irrep}
\end{array}$$

and get induced maps

$$W_\alpha \to W \xrightarrow{\Phi_W} W \to W_\alpha$$

for any inclusion of $W_\alpha \hookrightarrow W$.

But this composition must equal 0 if $\langle \Phi, \chi_W \rangle = 0 \ \forall$ irreps W. Hence Φ_W must equal 0, too. //