Class 6

Last time:

Prop'n $\{\chi_V\}_{V \text{ irrep}}$ is an orthonormal cllxn in $C(G)$.

$C(G) \subset \mathbb{C}[G]$, the cllxn of fxns on G constant along conj. classes

Today:

Prop'n

$$\operatorname{Span}\{\chi_V\}_{V \text{ irrep}} = C(G).$$

Before we go on:

Exer Let U, V be fin-dim G-reps (not nec. irreducible). Show

$$\langle \chi_U, \chi_V \rangle = \dim\left(\hom_G(U,V)\right).$$

Recall: $\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$

Exer Suppose

$$U \cong \bigoplus_\alpha V_\alpha^{n_\alpha}$$

where α runs over irreps V_α.

To get started, try showing $\hom_G(V \oplus V, V \oplus V) \cong M_{2\times 2}(k)$ when V is irrep.

Show

$$\operatorname{End}_G(U) \cong \prod_\alpha M_{n_\alpha \times n_\alpha}(\mathbb{C}) \quad \text{as rings.}$$

G-linear endomorphisms

$n_\alpha \times n_\alpha$ matrices w/ $\mathbb{C}$ entries

Sol'n Same as last time:

$$\langle \chi_u, \chi_v \rangle = \frac{1}{|G|} \sum \chi_u(g)\overline{\chi_v(g)}$$

$$= \frac{1}{|G|} \sum \chi_{V^\vee \otimes U}(g)$$

$$= \dim\left(\hom(V,U)^G\right)$$

$$= \dim\left(\hom_G(V,U)\right).$$

Note that

$\langle \chi_v, \chi_u \rangle = \overline{\langle \chi_u, \chi_v \rangle}$ by defn

hence, knowing $\langle \chi_u, \chi_v \rangle \in \mathbb{R}$,

have

$$\langle \chi_v, \chi_u \rangle = \langle \chi_u, \chi_v \rangle$$

hence

$$\begin{array}{ccc} \dim(\hom_G(V,U)) & & \dim(\hom_G(U,V)) \\ \shortparallel & & \shortparallel \\ \langle \chi_u, \chi_v \rangle & = & \langle \chi_v, \chi_u \rangle \end{array}$$

Cor For U, V G-reps,

$$\hom_G(U,V) = \hom_G(V,U).$$

Rmk Not surprising given Schur's Lemma and complete reducibility: $\hom(V,U)$, $\hom(U,V)$ both only depend on the irrep. factors of U and of V.

Also, last time I didn't prove the following in class:

<u>Lemma</u> For any G-rep V,

$$\frac{1}{G} \sum_g \chi_V(g) = \dim V^G.$$

<u>Pf</u>: Define the linear function

$$\begin{aligned} \pi : V &\longrightarrow V \\ v &\longmapsto \frac{1}{|G|} \sum_g gv. \end{aligned}$$

Note: (a) $h\pi(v) = \pi(v) \ \forall h \in G$:

$$\begin{aligned} h\pi(v) &= \frac{1}{|G|} \sum hgv \\ &= \frac{1}{|G|} \sum_y yv \\ &= \pi(v) \end{aligned}$$

$$\begin{array}{cc} G = \{g\} & g \\ \wr\!\!\downarrow & \updownarrow \\ G = \{y\} & hg \end{array}$$

(b) $v \in V^G \Rightarrow \pi(v) = v$

$$\pi(v) = \frac{1}{|G|} \cdot |G| v = v.$$

So $\pi^2 = \pi$ (ie, π is a projection).

Hence in some basis, $\pi = \begin{pmatrix} I & \\ & 0 \end{pmatrix}$

and $\operatorname{trace}(\pi) = \dim V^G$.

OTOH, $\operatorname{trace}(\pi) = \frac{1}{|G|} \sum \operatorname{trace}(g) = \frac{1}{|G|} \sum \chi_V(g)$. //

Now, for the other "exercice."

Recall the univ prop of $U \oplus V$
(as vec spaces; no G-rep yet).

$$\begin{tikzcd}
 & U \arrow[dr, "f_U"] & \\
V \arrow[rr, "f_V"'] & & W
\end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd}
 & U \arrow[d, "i_U"] \arrow[ddr, bend left, "f_U"] & \\
V \arrow[r, "i_V"] \arrow[drr, bend right, "f_V"'] & U \oplus V \arrow[dr, dotted, "\exists! j"] & \\
 & & W
\end{tikzcd}$$

$\exists!\, j$ s.t. this diagram commutes

This gives a fuction

$$\hom_k(U,W) \times \hom_k(V,W) \longrightarrow \hom(U \oplus V, W)$$
$$(f_U, f_V) \longmapsto j$$

which has an inverse:

$$(j \circ i_U,\ j \circ i_V) \longleftarrow\!\!\shortmid\ j$$

ie, $\hom(U,W) \times \hom(V,W) \cong \hom(U \oplus V, W)$.

Can check this isom. is linear.

Though $U \times V = U \oplus V$, let me now obcess over $U \times V$.

$U \times V$ has projections

$$\begin{tikzcd} U \times V \arrow[r, "p_U"] \arrow[d, "p_V"'] & U \\ V & \end{tikzcd} \qquad \begin{tikzcd} (u,v) \arrow[r, mapsto] \arrow[d, mapsto] & u \\ v & \end{tikzcd}$$

and has the following univ. prop.:

$\forall$ linear fxns

$$\begin{tikzcd} W \arrow[r, bend left, "f_U"] \arrow[d, bend right, "f_V"'] & U \\ V & \end{tikzcd}$$

$\exists !$ $j : W \longrightarrow U \times V$ st

$$\begin{tikzcd} W \arrow[dr, "j"] \arrow[drr, bend left, "f_U"] \arrow[ddr, bend right, "f_V"'] & & \\ & U \times V \arrow[r, "p_U"'] \arrow[d, "p_V"] & U \\ & V & \end{tikzcd}$$ commutes.

$\Big(j(w) = (f_U(w), f_V(w)) \Big).$

This gives a linear isomorphism

$$\hom(W, U) \times \hom(W, V) \longrightarrow \hom(W, U \times V)$$
$$(f_U, f_V) \longmapsto j$$

with inverse $j \mapsto (p_U \circ j, p_V \circ j)$.

Then, when the indexing set $\{\beta\}$ is finite, have

$$\hom\Big(\bigoplus_\alpha U_\alpha, \bigoplus_\beta V_\beta\Big)$$

$$\cong \quad \beta \text{ finite}$$

$$\hom\Big(\bigoplus_\alpha U_\alpha, \prod_\beta V_\beta\Big)$$

$$\cong \quad \text{univ prop of } \oplus$$

$$\prod_\alpha \hom\Big(U_\alpha, \prod_\beta V_\beta\Big)$$

$$\cong \quad \text{univ prop of } \times \text{ (i.e. } \textstyle\prod\text{)}$$

$$\prod_{\alpha,\beta} \hom(U_\alpha, V_\beta).$$

⚠ Have not yet defined $\bigoplus_\alpha V_\alpha$ for $\{\alpha\}$ infinite. But here it is:

$$\bigoplus_\alpha V_\alpha = \Big\{ \text{tuples } (v_\alpha)_\alpha,\ v_\alpha \in V_\alpha, \text{ where only finitely many } v_\alpha \text{ are non-zero.} \Big\} \subset \prod_\alpha V_\alpha.$$

Rmk In general, have

$$\hom\Big(\bigoplus_\alpha U_\alpha, \prod_\beta V_\beta\Big) \cong \prod_{\alpha,\beta} \hom(U_\alpha, V_\beta) \qquad (\star)$$

regardless of indexing sets.

Exer (On your own time:)

$(\star)$ is true for G-reps:

$$\hom_G\Big(\bigoplus_\alpha U_\alpha, \prod_\beta V_\beta\Big) \cong \prod_{\alpha,\beta} \hom_G(U_\alpha, V_\beta).$$

(Not hard; same formulas.)

These isomorphisms are <u>natural</u>, in that if we have maps

$$\phi_\alpha : U_\alpha' \longrightarrow U_\alpha$$
$$\psi_\beta : V_\beta \longrightarrow V_{\beta'} ,$$

$$\begin{tikzcd}
\hom\left(\bigoplus_\alpha U_\alpha, \prod_\beta V_\beta\right) \arrow[r, phantom, "\cong"] \arrow[d, "\psi\circ\phi"'] & \prod_{\alpha,\beta} \hom(U_\alpha, V_\beta) \arrow[d, "\psi\circ\phi"] \\
\hom\left(\bigoplus_\alpha U_\alpha', \prod_\beta V_\beta'\right) \arrow[r, phantom, "\cong"] & \prod_{\alpha,\beta} \hom(U_\alpha', V_\beta')
\end{tikzcd}$$

commutes.

<u>Example</u> Fix irrep V. Then

$$\hom(V\oplus V, V\oplus V) \cong ?$$

For sanity, label as $\hom(V_1\oplus V_2, V_1\oplus V_2)$.
Then by univ. prop. and $V\oplus V = V\times V$, have

$$\begin{aligned}
&\hom(V_1\oplus V_2, V_1\oplus V_2)\\
&\qquad \cong\\
&\hom(V_1,V_1)\times\hom(V_1,V_2)\\
\times\ &\hom(V_2,V_1)\times\hom(V_2,V_2)\\
&\qquad \cong\\
&\quad k \times k\\
\times\ &\quad k \times k.\\
&\qquad \cong\\
&\{(a_{ij}) \mid a_{ij}\in k,\ i,j=1,2\}.
\end{aligned}$$

(suggestively writing in matrix form)

Naturality gives composition law:

$$\begin{tikzcd}
V_1 \arrow[r, "a_{11}"] \arrow[dr, "a_{21}"'] & V_1 \arrow[r, "b_{11}"] \arrow[dr, "b_{21}"] & V_1 \\
\oplus & \oplus & \oplus \\
V_2 \arrow[r, "a_{22}"'] \arrow[ur, "a_{12}"] & V_2 \arrow[r, "b_{22}"'] \arrow[ur, "b_{12}"] & V_2
\end{tikzcd}$$

$\big\downarrow$ compose

$$\begin{tikzcd}
V_1 \arrow[r, "b_{11}a_{11} + b_{12}a_{21}"] \arrow[dr, "b_{21}a_{11} + b_{21}a_{12}"] & V_1 \\
\oplus & \oplus \\
V_2 \arrow[r, "b_{22}a_{22} + b_{21}a_{12}"'] \arrow[ur, dotted] & V_2
\end{tikzcd}$$

this <u>is</u> matrix multiplication.

More generally (finally getting to original exercise):

$$\mathrm{End}_G(V, V)$$
$$\cong$$
$$\mathrm{End}_G\left(\bigoplus_\alpha V_\alpha^{n_\alpha}, \bigoplus_\alpha V_\alpha^{n_\alpha}\right)$$
$$\cong \qquad \alpha \text{ finite}$$
$$\hom_G\left(\bigoplus_\alpha V_\alpha^{n_\alpha}, \prod_\alpha V_\alpha^{n_\alpha}\right)$$
$$\cong$$
$$\prod_{\alpha_1, \alpha_2} \hom_G\left(V_{\alpha_1}^{\oplus n_{\alpha_1}}, V_{\alpha_2}^{n_{\alpha_2}}\right) \qquad n_{\alpha_1} \text{ finite}$$
$$\cong \quad \text{Schur's lemma: } 0 \text{ unless } \alpha_1 = \alpha_2$$
$$\prod_\alpha \hom_G\left(V_\alpha^{\oplus n_\alpha}, V_\alpha^{n_\alpha}\right)$$
$$\cong$$
$$\prod_\alpha \left(k^{n_\alpha}\right) \text{ as vec spaces, at least.}$$
$$\cong$$
$$\prod_\alpha M_{n_\alpha, n_\alpha}(k) \text{ by naturality.}$$

Now, to exhibit ring isom

$$\mathbb{C}G \longrightarrow \prod_\alpha M_{n_\alpha \times n_\alpha}(\mathbb{C}),$$

we use some basic facts

about rings and modules.

Recall:

<u>Defn</u> R a unital ring, not nec. commutative. A left R-module M is an abelian gp M, equipped w/ a ring homomorphism

$$R \longrightarrow \mathrm{End}(M).$$ ← abelian gp endomorphisms

If R is a k-algebra (meaning R is a k-module and $R \times R \to R$ is k-bilinear) we demand M be a k-mod, and

$$R \longrightarrow \mathrm{End}_k(M)$$ ← k-linear endomorphisms.

be k-linear.

<u>Rmk</u> Concretely, this gives rise to a fxn

$$R \times M \longrightarrow M$$

st

- $r(ax_1 + x_2) = arx_1 + rx_2$
- $r(sx) = (rs)x$
- $(ar+s)x = arx + sx$.

Defn A map of R-modules, or an R-module homom, is a (k-linear) fxn

— also called "an R-linear map"

$$\phi: M \longrightarrow N$$

st

$$\phi(rx) = r\phi(x).$$

Example Let $\mathbb{C}G$ be the group ring. Then a left $\mathbb{C}G$-module is a $\mathbb{C}$-vec space equipped w/ fxns

$$\mathbb{C}G \times V \longrightarrow V$$

st

$$\left(\left(\sum a_g g\right)\left(\sum b_h h\right)\right)v = \left(\sum a_g g\right)\left(\left(\sum b_h h\right)v\right).$$

By $\mathbb{C}$-linearity, this reduces to

$$(gh)v = g(hv).$$

ie, a left $\mathbb{C}G$-module is a G-representation, and vice versa.

Example R is a left module over itself:

$$R \times R \longrightarrow R$$

is multiplication.

Example When $R = \mathbb{C}G$, this left module R is the regular rep.

Prop

$\mathrm{End}_R(R) := \hom_R(R,R)$ ← R-module homom from R to itself.

$\cong R^{op}$. ← opposite algebra, where $r \cdot_{op} s = s \cdot r$.

Pf: Consider

$$\begin{aligned} R^{op} &\longrightarrow \mathrm{End}_R(R) \\ r &\longmapsto (\phi_r : x \mapsto xr) \end{aligned}$$

ϕ_r is R-linear because $\phi_r(sx) = (sx\, r)$

$$= s(xr)$$

$$= s\phi_r(x).$$

OTOH, any elmt $\phi \in \mathrm{End}_R(R)$ is determined completely by $\phi(1_R)$:

$$\phi(x) = \phi(x 1_R) = x\phi(1_R).$$

Finally,

$$\begin{aligned} \phi_{r_1 r_2}(x) &= x(r_1 r_2) \\ &= (x r_1) r_2 \\ &= \phi_{r_2} \circ \phi_{r_1}. \end{aligned}$$

So this is a ring isom. //

Cor $\mathbb{C}G^{op} \cong \prod_\alpha M_{n_\alpha \times n_\alpha}(\mathbb{C})$.

Rmk $M_{n_\alpha \times n_\alpha}(k) \cong M_{n_\alpha \times n_\alpha}(k)^{op}$, so this gives

$\mathbb{C}G \cong \prod_\alpha M_{n_\alpha \times n_\alpha}(k)$ and $\mathbb{C}G \cong \mathbb{C}G^{op}$.

Defn The center

$$Z(R)$$

of a ring R is the cllxn of $x \in R$ s.t. $\forall y \in R$, $xy = yx$.

Cor

$$Z(\mathbb{C}G) \cong Z\left(\prod_\alpha M_{n_\alpha \times n_\alpha}(k)\right)$$

$$\cong \prod_\alpha k$$

b/c center of matrix ring is the diagonal matrices, with constant diagonal entries.

In particular,

$$\dim_k(Z(\mathbb{C}G)) = \# \text{ irreps of } G.$$

We are now left to prove:

Propn

$$Z(\mathbb{C}G) = \{\text{class fxns}\}.$$