Lecture 7

Notes from class may differ from written notes.

Last time, stated:

<u>Lemma</u> ① $\mathrm{End}_R(R) \cong R^{op}$

<u>Lemma</u> ② $\mathrm{End}_G(V_{reg}) \cong \prod_\alpha M_{n_\alpha \times n_\alpha}(\mathbb{C})$

where $\{\alpha\}$ runs over irreps V_α of G,

$n_\alpha = \dim(V_\alpha)$

<u>Lemma</u> ③ $Z(\mathbb{C}G) = C(G) = \{\text{class fns}\}$.

Showed Lemmas $\Longrightarrow$ $C(G) \cong k^{\# \frac{G}{G}}$

$\Longrightarrow$ # irreps = # $\frac{G}{G}$.

Also proved Lemma ①.

Today: prove Lemmas ②, ③.

<u>Defn</u> Fix a cllxn of vector spaces $\{V_\alpha\}$ where $\{\alpha\} =: A$. Then the direct product of the V_α is defined to be the vector space

$$\prod_{\alpha \in A} V_\alpha := \{ (v_\alpha)_{\alpha \in A} \mid v_\alpha \in V_\alpha \}.$$

The direct sum of $\{V_\alpha\}$ is the vector space

$$\bigoplus_{\alpha \in A} V_\alpha \subset \prod_{\alpha \in A} V_\alpha$$

of those (v_α) s.t. only finitely many v_α are non-zero.

<u>Rmk</u> When $\{\alpha\}$ finite, $\bigoplus V_\alpha \hookrightarrow \prod V_\alpha$ is an isomorphism.

Note that there are projections

$$\prod_\alpha V_\alpha \xrightarrow{p_\alpha} V_\alpha \quad , \quad (v_\alpha) \longmapsto v_\alpha$$

and inclusions

$$V_\alpha \xhookrightarrow{i_\alpha} \bigoplus_\alpha V_\alpha \qquad v_\alpha \longmapsto (v_\beta) \text{ where } v_\beta = \begin{cases} v_\alpha & \alpha = \beta \\ 0 & \text{otherwise} \end{cases}$$

The projections and inclusions induce maps (for any W)

$$\hom(W, \prod_\alpha V_\alpha) \longrightarrow \hom(W, V_\alpha)$$
$$f \longmapsto p_\alpha \circ f$$

$$\hom(\bigoplus_\alpha V_\alpha, W) \longrightarrow \hom(V_\alpha, W)$$
$$g \longmapsto g \circ i_\alpha$$

hence maps

$$\hom(W, \prod_\alpha V_\alpha) \longrightarrow \prod_\alpha \hom(W, V_\alpha)$$
$$f \longmapsto (p_\alpha \circ f)_\alpha$$

$$\hom(\bigoplus_\alpha V_\alpha, W) \longrightarrow \prod_\alpha \hom(V_\alpha, W)$$
$$g \longmapsto (g \circ i_\alpha)_\alpha$$

these maps are both isomorphisms, and natural. In English, these isomorphisms mean:

"To give maps $\{f_\alpha : W \to V_\alpha\}$ is the same as giving a single map $f : W \to \prod_\alpha V_\alpha$."

"To give maps $\{g_\alpha : V_\alpha \to W\}$ is the same as giving a single map $g : \bigoplus_\alpha V_\alpha \longrightarrow W$."

Now suppose $A = \{\alpha\}$ is finite, so $\bigoplus V_\alpha = \prod V_\alpha$.

Then we have

$$\hom\left(\bigoplus_\alpha V_\alpha, \prod_\alpha V_\alpha\right) = \hom\left(\bigoplus V_\alpha, \bigoplus V_\alpha\right) =: \mathrm{End}(V_\alpha)$$

$$\cong$$

$$\prod_\alpha \hom\left(V_\alpha, \prod_\beta V_\beta\right)$$

writing $A = \{\alpha\} = \{\beta\}$ for sanity's sake.

$$\cong$$

$$\prod_{\alpha,\beta} \hom(V_\alpha, V_\beta)$$

But End(anything) is always a ring because endomorphisms compose. How do we read off the ring structure in $\prod_{\alpha,\beta} \hom(V_\alpha, V_\beta)$?

By naturality, composition is given as follows: if $(\phi_{\alpha\beta})_{\alpha,\beta} \in \prod_{\alpha,\beta} \hom(V_\alpha, V_\beta)$,

$$(\psi_{\beta\gamma})_{\beta,\gamma} \in \prod_{\beta,\gamma} \hom(V_\beta, V_\gamma),$$

then

$$(\psi_{\beta\gamma})_{\beta,\gamma} \circ (\phi_{\alpha\beta})_{\alpha,\beta} = \left(\sum_\beta \psi_{\beta\gamma} \circ \phi_{\alpha\beta}\right)_{\alpha,\gamma} \in \prod_{\alpha,\gamma} \hom(V_\alpha, V_\gamma).$$

(Summation is finite by assumption on A.)

"The (α,γ) entry is the sum of $\psi_{\beta\gamma}\phi_{\alpha\beta}$ over all β."

This is just a generalized version of a matrix ring.

Now:

Pf (of Lemma ②)

$$\mathrm{End}\left(\oplus V_\alpha^{n_\alpha}, \oplus V_\alpha^{n_\alpha}\right) \cong \prod_{\alpha,\beta} \hom\left(V_\alpha^{n_\alpha}, V_\beta^{n_\beta}\right)$$

univ prop. (i.e., discussion so far)

$$\cong \prod_{\alpha=\beta} \hom\left(V_\alpha^{n_\alpha}, V_\beta^{n_\beta}\right) \times \prod_{\alpha\neq\beta} \{0\}$$

Schur's Lemma

$$\cong \prod_\alpha \mathrm{End}\left(V_\alpha^{n_\alpha}\right)$$

$$\cong \prod_\alpha \left(\hom\left(\bigoplus_{i=1}^{n_\alpha} V_\alpha, \bigoplus_{j=1}^{n_\alpha} V_\alpha\right)\right)$$

$$\cong \prod_\alpha \left(\prod_{i,j} \hom\left(V_\alpha, V_\alpha\right)\right)$$

$$\cong \prod_\alpha \left(\prod_{i,j} \mathbb{C} \right)$$

Schur's Lemma

$$\cong \prod_\alpha M_{n_\alpha \times n_\alpha}(\mathbb{C}) \quad //$$

since by naturality,

$$(b_{jk})(a_{ij})_{ik} = \sum_j b_{jk} a_{ij}.$$

To prove Lemma ③, observe:

Prop'n Fix unital ring R. and $r \in R$.

TFAE:

(i) $\phi_r : R \longrightarrow R$, $x \longmapsto rx$ is a map of left R-modules

(ii) For every left R-module M,

$$\phi_r : M \longrightarrow M, \quad m \longmapsto rm$$

is a map of left R-modules

(iii) $\forall\, x \in R$, $xr = rx$.

Rmk (iii) means $r \in Z(R)$,

ie, r is in the center of R.

Pf (iii) $\Rightarrow$ (ii): $\phi_r(xm) = r(xm) = (rx)m = (xr)m = x(rm) = x\phi_r(m)$.

(ii) $\Rightarrow$ (i): obvious.

(i) $\Rightarrow$ (iii): ϕ_r module map $\Rightarrow$ $x\phi_r(1) = \phi_r(x \cdot 1)$. ie, $xr = rx$. //

Rmk There's something "natural" about description (ii): it defines a map for any M, and moreover, for any morphism $\psi : M \to M$, can check

$$\begin{tikzcd} M \arrow[r, "\phi_r"] \arrow[d, "\psi"'] & M \arrow[d, "\psi"] \\ M \arrow[r, "\phi_r"'] & M \end{tikzcd}$$

commutes. This leads to a notion of "center" for any category.

Now assume a fxn

$$f: G \longrightarrow \mathbb{C}$$
$$g \longmapsto f(g)$$

(ie, an element $\sum_g f(g) e_g \in \mathbb{C}G$)

defines a module map of $V_{reg} = \mathbb{C}G$.

Then have:

$$\left(\sum_g f(g) e_g\right)(e_h) = h\left(\sum f(g) e_g\right).$$

i.e.,

$$\sum_g f(g)\, e_{gh} = \sum_g f(g)\, e_{hg}.$$

Multiply by h^{-1}:

$$\sum f(g) e_g = \sum f(g)\, e_{hgh^{-1}}. \quad //$$