Lecture 8)

We finally get to character tables:

Defn Fix a finite group G. Its <u>character table</u> is a $\#\frac{G}{G}$-by-$\#\frac{G}{G}$ array:

# of elts in $[g]$

conj class in G

$$\begin{array}{c|ccc} & \cdots & \overset{\#[g]}{[g]} & \cdots \\ \hline \vdots & & & \\ V & & \chi_V(g) & \\ \vdots & & & \end{array}$$

an irrep

character of irrep, evaluated at $g \in [g]$.

Rmk Since $\#\{\text{irreps}\}/_{\text{isom}} = \#\frac{G}{G}$, we know this is a square array.

Rmk This is a convenient way to record the data of the irreps of G, and is presented in a way that makes <u>completing</u> the table easier, too.

Ex $G = C_N = \mathbb{Z}/N\mathbb{Z}$.

Since G is abelian,

$$\begin{aligned} G &\longrightarrow \frac{G}{G} \quad \text{is a bijection.} \\ g &\longmapsto [g] \end{aligned}$$

Prop'n Any irrep of an abelian gp G is 1-dim.

Pf Note $\mathbb{C}G$ is commutative if G is abelian - hence the center $Z(\mathbb{C}G)$ is $\mathbb{C}G$ itself. In particular, for any $g \in G$, $g : V \longrightarrow V$ is a map of G-reps. By Schur's lemma, if V is an irrep, g acts by scaling. Since any $g \in G$ acts by scaling, any subspace $V' \subset V$ is a subrepresentation; so $\dim V = 1$. //

Enumerating the irreps by $V_0, \dots, V_N$, we thus have

	1	1		1
	$[e]$	$[1]$	$[2] \cdots$	$[N-1]$
$V_{triv} = V_0$	1	1	1 $\cdots$	1
V_1	1			
$\vdots$	$\vdots$			
V_{N-1}	1			

b/c $\text{trace}(id_{1\times 1}) = 1$.

because $\chi_V(id_G) = \dim V$.

On the other hand, we know

$$\begin{aligned} C_N &\xrightarrow{\chi_a} \mathbb{C}^\times = GL_1(\mathbb{C}) \\ 1 &\longmapsto e^{2\pi i a/N} \end{aligned}$$

is a gp homom, and there are precisely N of these, given by $a = \{0, 1, \cdots, N-1\}$. This completes the character table for C_N:

	1 $[0]$	1 $[1]$	$\cdots$	$\cdots$ 1 $[b]$	$\cdots$
V_{triv}	1	1	$\cdots$		
$\vdots$		$\vdots$			
V_a	1	$\chi_{V_a}(1) = e^{2\pi a i/N}$	$\cdots$	$e^{2\pi a b i/N}$	
$\vdots$					

Ex Set $\alpha = e^{2\pi i/3}$.

	1 $[0]$	1 $[1]$	1 $[2]$
V_{triv}	1	1	1
V_1	1	α	α^2
V_2	1	α^2	α

depending on your familiarity w/ identities like this this is cool.

In genl, center of mass of N^{th} roots of unity is 0.

Orthonormality of characters:

$\langle \chi_{triv}, \chi_{triv} \rangle = \frac{1}{3}(1+1+1) = 1$ $\qquad$ $\langle \chi_{triv}, \chi_2 \rangle = \frac{1}{3}(1 + \bar{\alpha} + \bar{\alpha}^2) = 0.$

$\langle \chi_1, \chi_1 \rangle = \frac{1}{3}(1 + \alpha\bar{\alpha} + \alpha^2\bar{\alpha}^2) = 1.$ $\qquad$ $\langle \chi_1, \chi_2 \rangle = \frac{1}{3}(1 + \alpha\bar{\alpha}^2 + \alpha^2\bar{\alpha}) = 0.$

$\langle \chi_2, \chi_2 \rangle = 1.$

Ex $G = S_3$. Recall that for symmetric gps, conj. classes are given by cycle shapes: in this case,

$$[e] = [(1)] = \{e\}$$

$$[(12)] = \{(12), (23), (13)\}$$

$$[(123)] = \{(123), (132)\}.$$

As for irreps, we have

- trivial rep, and
- sign rep: $S_n \to \mathbb{C}^\times$, $\sigma \mapsto (-1)^\sigma \in \{\pm 1\}$ ← irrep b/c $\dim V = 1$.

So we have:

	1 $[e]$	3 $[(12)]$	2 $[123]$
V_{triv}	1	1	1
V_{sign}	1	-1	1
?	?	?	?

Note that orthonormality so far confirms:

$$\langle \chi_{triv}, \chi_{sign} \rangle = \frac{1}{6}\left(1 + 3(1 \cdot -1) + 2(1 \cdot 1)\right)$$
$$= \frac{1}{6}(1 - 3 + 2) = 0.$$

$$\langle \chi_{sign}, \chi_{sign} \rangle = \frac{1}{6}\left(1 \cdot 1 + 3(-1 \cdot -1) + 2(1 \cdot 1)\right)$$
$$= \frac{1}{6} 6$$
$$= 1.$$

this is why we record $\#[g]$ in character tables; easy to check/compute $\langle , \rangle$!

At this point, we know:

- $\exists$ one more irrep (because $\# \frac{G}{\sim} = 3$)

- It must have dim 2. (because $1+1+d^2=6 \Rightarrow d=2$.)

Hence have

	1 [e]	3 [(12)]	2 [(123)]
V_{triv}	1	1	1
V_{sign}	1	-1	1
V	2	s	t

by orthonormality:

$$0 = \langle \chi_{triv}, \chi_V \rangle = \frac{1}{6}\left(1 \cdot 2 + 3\bar{s} + 2\bar{t}\right)$$

$$\Rightarrow 3\bar{s} + 2\bar{t} = -2.$$

$$0 = \langle \chi_{sign}, \chi_V \rangle = \frac{1}{6}\left(1 \cdot 2 - 3\bar{s} + 2\bar{t}\right)$$

$$\Rightarrow -3\bar{s} + 2\bar{t} = -2.$$

$$\Rightarrow 6\bar{s} = 0 \Rightarrow \bar{s} = 0$$

$$\Rightarrow t = -1.$$

This completes charac. table.

Rmk Charac. table doesn't produce representations for you; you sometimes need insight or context to be able to realize these characters by specific representations.

In this case, we have that

$$S_3 \circlearrowright \{1,2,3\}$$

by defn, hence

$$S_3 \circlearrowright \mathbb{C}\langle\{1,2,3\}\rangle = \mathbb{C}^3.$$

The subspace $\{a(e_1+e_2+e_3)\}_{a\in\mathbb{C}} \subset \mathbb{C}^3$ is a 1-dim subrep, $\cong$ to V_{triv}.

Hence $\mathbb{C}^3 \cong V_{triv} \oplus V'$

and one can check $V' \cong V$ (V is the irrep in the character table) by seeing

$$\chi_{V'} = \chi_V.$$

For instance: set $f_1 = e_1 - e_2$

$f_2 = e_2 - e_3$.

$(f_1 + f_2 = e_1 - e_3)$.

[To see matrix of (123) in f_1, f_2 basis:
$(123)f_1 = e_2 - e_3 = f_2$
$(123)f_2 = e_3 - e_1 = -f_1 - f_2$]

Then
$$\left.\begin{aligned} (12)\cdot f_1 &= -f_1 \\ (12)\cdot f_2 &= f_2 \\ (13)f_1 &= -f_2 \\ (13)f_2 &= -f_1 \\ (23)f_1 &= f_1+f_2 \\ (23)f_2 &= -f_2 \end{aligned}\right\}$$
so $\mathrm{span}\langle f_1, f_2\rangle =: V' < \mathbb{C}^3$ is a subrep.

$\chi_{V'}((12)) = \mathrm{trace}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = 0,$

while $\chi_{V'}((123)) = \mathrm{trace}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = -1.$ $\implies \chi_{V'} = \chi_V \implies V' \cong V.$

Now, the cnstrxn

$$V \longmapsto V \otimes V$$

is often a fruitful way to make new representations.

In homework, you showed

$$\underset{C_2}{\overset{}{V \otimes V}} \cong \{+1 \text{ eigenspace}\} \oplus \{-1 \text{ eigenspace}\}.$$

Concretely: If $\{e_i\}_{i \in I}$ is a basis for V, have

$$\mathrm{Span}\{e_i \otimes e_i\} \circlearrowleft +1$$
$$\oplus$$
$$\mathrm{Span}\{e_i \otimes e_j - e_j \otimes e_i\} \circlearrowleft -1$$
$$\cong$$
$$V \otimes V.$$

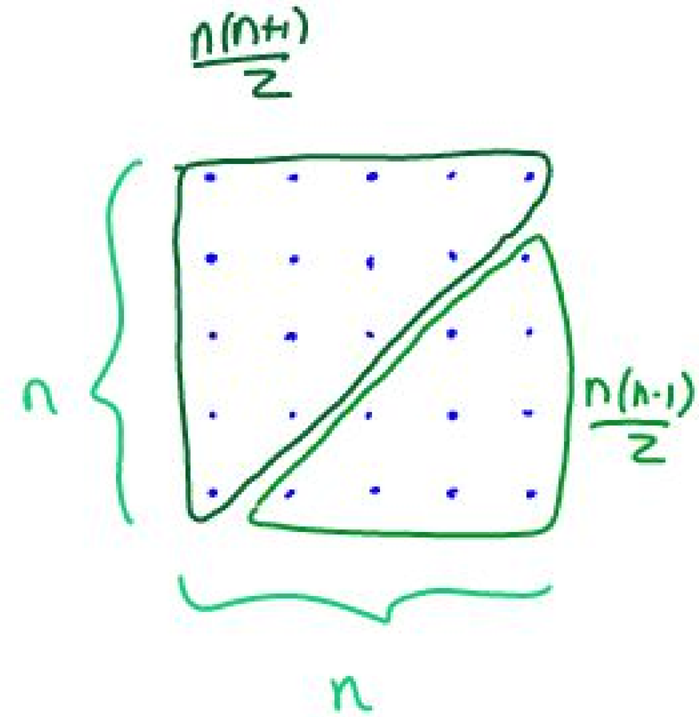
<u>Defn</u> $\mathrm{Sym}^2 V := \mathrm{Span}\{e_i \otimes e_j + e_j \otimes e_i\}_{i \le j}$ $\supset \mathrm{Span}\{e_i \otimes e_i\}$ since char $k \neq 2$.

$\Lambda^2 V := \mathrm{Alt}^2 V := \mathrm{Span}\{e_i \otimes e_j - e_j \otimes e_i\}_{i<j}$.

are the <u>2^{nd} symmetric power</u>, and <u>2^{nd} alternating power</u> of V, respectively.

<u>Rmk</u> $\dim \mathrm{Sym}^2 V = \frac{n(n+1)}{2}$

$\dim \mathrm{Alt}^2 V = \frac{n(n-1)}{2}$.



Propn

$$\chi_{\Lambda^2 V}(g) = \frac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right).$$

$$\chi_{\mathrm{Sym}^2 V}(g) = \frac{1}{2}\left(\chi_V(g)^2 + \chi_V(g^2)\right).$$

Pf Let $\{e_i\}$ be an eigenbasis for $g \in G$ w/ eigenvalues $\{\lambda_i\}$. Then

$$g(e_i \otimes e_j - e_j \otimes e_i) = \lambda_i \lambda_j (e_i \otimes e_j - e_j \otimes e_i).$$

Hence

$$\chi_{\Lambda^2 V}(g) = \sum_{i<j} \lambda_i \lambda_j$$

$$= \frac{1}{2}\left(\left(\sum_i \lambda_i\right)^2 - \sum_i (\lambda_i)^2\right)$$

$$= \frac{1}{2}\left((\chi_V(g))^2 - \chi_V(g^2)\right).$$

Likewise, since $g(e_i \otimes e_j + e_j \otimes e_i) = \lambda_{ij}(e_i \otimes e_j + e_j \otimes e_i)$, we have

$$\chi_{\mathrm{Sym}^2(V)}(g) = \sum_{i \le j} \lambda_i \lambda_j$$

$$= \frac{1}{2}\left(\left(\sum \lambda_i\right)^2 + \sum \lambda_i^2\right).$$

$\underline{\underline{\text{Ex}}}$ $G = S_4$.

What are the conjugacy classes?

$[(1)] = \{e\}$

$[(12)] = \{(12),(13),(14)$
$(23),(24),(23)\}$

$[(12)(34)] = \{(12)(34),(13)(24),(14)(23)\}$

$[(123)] = \{(123),(132),(124),(142),(134),(143),$
$(234),(243)\}$

$[(1234)] = \{(1234),(1243),(1324),$
$(1342),(1423),(1432)\}$

So we seek five irreps.

Note that since the abelianization of S_n is $\mathbb{Z}/2\mathbb{Z}$, we only have two irreps of dimension 1; V_{triv} and V_{sign}. We seek irreps V_a, V_b, V_c of dimensions n_a, n_b, n_c so that

$$1 + 1 + n_a^2 + n_b^2 + n_c^2 = 24.$$
$$= |S_4|.$$

What triple of squares adds to 22?

Only $(3^2, 3^2, 2^2)$. So char. table must look like:

	1 e	6 (12)	8 (123)	3 (12)(34)	6 (1234)
V_{triv}	1	1	1	1	1
V_{sign}	1	-1	1	1	-1
V_a	2				
V_b	3				
V_c	3				

It turns out we can always find an irrep of dimension $n-1$:

<u>Prop</u> The action $S_n \curvearrowright \mathbb{C}^n$ always splits as

$$\mathbb{C}^n \cong V_{triv} \oplus V$$

where V is irreducible.

<u>Def</u> V is called the <u>standard</u> representation.

<u>Pf</u> See homework.

<u>Rmk</u> The easiest way to define V_{std} is as the kernel of the map

$$\begin{aligned} \mathbb{C}^n &\longrightarrow \mathbb{C} \\ (a_i) &\longmapsto \sum a_i. \end{aligned}$$

<u>Rmk</u>:

We know

$$\chi_{\mathbb{C}^n}(g) = \#\mathrm{Fix}(g),$$

where $\mathrm{Fix}(g) \subset \{1, \cdots, n\}$ is the fixed pt set of g. Hence

$$\chi_{V_{std}}(g) = \chi_{\mathbb{C}^n}(g) - \chi_{triv}(g) = \#\mathrm{Fix}(g) - 1.$$

It's not so obvious that

$\langle \chi_{V_{std}}, \chi_{V_{std}} \rangle$ (which involves summing the products "sizes of conj. classes" · "length of cycles = #Fix") equals 1

	1 e	6 (12)	8 (123)	3 $(12)(34)$	6 (1234)
V_{triv}	1	1	1	1	1
V_{sign}	1	-1	1	1	-1
V_a	2				
V_{std}	3	1	0	-1	-1
V_c	3				

<u>Check</u>: $\|\chi_{std}\|^2$

$$= \frac{1}{24}\left(1\cdot 3^2 + 6\cdot 1^2 + 8\cdot 0 + 3\cdot(-1)^2 + 6\cdot(-1)^2\right).$$

$$= \frac{1}{24}(9+6+3+6) = 1.$$

Now, since all characters so far are real, we cheaply make a new irrep by tensoring V_{std} w/ V_{sign}:

$$V_c := V_{sign} \otimes V_{std}. \qquad \Rightarrow \chi_{V_c} \cdot \chi_{V_{std}}$$

Now, we can find the character for the final V_a using orthogonality of characters.

$$\langle V_a, V_{std} \rangle = 0 = \frac{1}{24}\left(6 + 6A + 8 \cdot 0 \cdot B - 3 \cdot \underline{1} \cdot C - 6 \cdot \underline{1} \cdot D\right)$$

$$\Rightarrow -6A + 3C + 6D = 6.$$

$$\langle V_a, V_c \rangle = 0 = \frac{1}{24}\left(1 \cdot 6 + 6(-1)A + 3 \cdot (-1) \cdot C + 6 \underline{1} \cdot D\right)$$

$$\Rightarrow 6 = 6A + 3C - 6D$$

$$\Rightarrow A = -D.$$

$$\Rightarrow 4D + C = 2, \quad -4D + C = 2 \Rightarrow D = 0, C = 2, A = 0.$$

$$\langle V_a, V_{triv} \rangle = 0 = \frac{1}{24}\left(2 + 6A + 8B + 3C + 6D\right)$$

$$\Rightarrow 2 + 8B + 6 = 0$$

$$\Rightarrow B = -1.$$

	1 e	6 (12)	8 (123)	3 $(12)(34)$	6 (1234)
V_{triv}	1	1	1	1	1
V_{sign}	1	-1	1	1	$-\underline{1}$
V_a	2	A	B	C	D
V_{std}	3	1	0	$-\underline{1}$	$-\underline{1}$
V_c	3	$-\underline{1}$	0	-1	$\underline{1}$

Can we guess what the representation V_a is?

Well, $(12)(34)$ has order 2, and acts on a 2-dim vec space w/ trace 2. Hence $(12)(34)$ (and its conjugates) act as the identity on V_a; we have

$$\begin{tikzcd} S_4 \arrow[r, "V_a"] \arrow[d] & GL_2(\mathbb{C}) \\ S_4/\{(12)(34), (13)(24), (14)(23), e\} \arrow[ur] & \end{tikzcd}$$

Exer $S_4/\{(12)(34), (13)(24), (14)(23)\} \cong S_3$. ← in fact, $S_4 \cong (C_2 \times C_2) \rtimes S_3$

So V_a irrep as S_4-rep

$\Longrightarrow V_a$ irrep as S_3-rep ← very special! V_a is a rep of S_4, restricted to a rep of the subgp S_3. Such restrictions of irreps are NOT always irreps. Here, S_3 is both a subgroup and a quotient of S_4.

But we saw S_3 has only one 2-dim irrep, its std rep.

	1 e	6 (12)	8 (123)	3 $(12)(34)$	6 (1234)
V_{triv}	1	1	1	1	1
V_{sign}	1	-1	1	1	-1
V_a	2	0	-1	2	0
V_{std}	3	1	0	-1	-1
V_c	3	-1	0	-1	1

Ex How does $V \otimes V$ decompose? (Here, not nec. for character table, just illustrating what we can do.)

$\chi_{\Lambda^2 V}(g) = \frac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right)$ so

	e	(12)	(123)	$(12)(34)$	(1234)	
χ_V	2	0	-1	2	0	
$\chi_{\Lambda^2 V}$	1	-1	1	1	-1	$\leftarrow \chi_{V_{sign}}$

while $\chi_{Sym^2(V)}(g) = \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right)$ means

$\chi_{Sym^2 V}$	3	1	0	3	1

this is NOT irreducible (since it's not on char table).

By inspection,

$$\chi_{Sym^2 V} = \chi_{triv} + \chi_{V_a}.$$

Hence

$$V \otimes V \cong V \oplus V_{triv} \oplus V_{sign}.$$

	1 e	6 (12)	8 (123)	3 $(12)(34)$	6 (1234)
V_{triv}	1	1	1	1	1
V_{sign}	1	-1	1	1	-1
V_a	2	0	-1	2	0
V_{std}	3	1	0	-1	-1
V_c	3	-1	0	-1	1