Lecture 10: Localizations

A word about the Hilbert basis theorem: The word "Hilbert" is there because Hilbert proved it. The word "basis" is in there because in the past, "basis" used to mean any finite spanning set. The Hilbert basis theorem says that any ideal of $R[x]$ admits a finite spanning set—i.e., one can find a basis for any ideal of $R[x]$.

The proof of this theorem apparently had a profound impact on math; it spurred the acceptance of non-constructive proofs. See file on the course website.

Anyhow, note the following corollaries:

COROLLARY 10.1. (1) Let R be Noetherian. Then $R[x_1, \ldots, x_n]$ is Noetherian. In particular, $k[x_1, \ldots, x_n]$ is Noetherian for any field k.
(2) Let R be Noetherian, and let S be an R-algebra which is finitely generated (as an algebra, not as a module). Then S is Noetherian.

PROOF. (1) We know $R[x_1]$ is Noetherian by the Hilbert basis theorem. Setting $R' = R[x_1, \ldots, x_{i-1}]$, we know that $R'[x_i]$ is Noetherian by the Hilbert basis theorem, so $R[x_1, \ldots, x_i]$ is Noetherian. The "in particular" follows because any field is Noetherian.

(2) For S to be finitely generated by an R-algebra means that S is the quotient of some polynomial algebra $R[x_1, \ldots, x_k]$. But the ascending chain condition is preserved under quotient maps. □

REMARK 10.2. Note the utility of playing off the two equivalent conditions: Ascending chains of ideals terminate, and every ideal being finitely generated. We now know that for any finitely generated R-algebra (R Noetherian), its ideals are finitely generated as R-modules (and hence as S-modules). This is also another utility of the notion of "module," as simultaneously generalizing ideals and vector spaces.

Another theorem about Noetherian things:

PROPOSITION 10.3. Let R be Noetherian, and M an R-module. If M is finitely generated, then M is Noetherian.

PROOF. We induct on the minimal number of generators of M. First, if M is generated by a single element x, the R-module map

$$R \to M, \qquad 1 \mapsto x$$

is a surjection, hence $M \cong R/I$. But submodules of R/I are in bijection with submodules of R containing I. So R Noetherian $\implies$ R/I is Noetherian.

The induction step is as follows: *Assume that for any module M' generated by k elements, every submodule of M' is finitely generated. Then the same is true for any module M generated by $k+1$ elements.*

Let M be generated by $x_1, \ldots, x_{k+1}$, and let N be an arbitrary submodule. We now play N off with a well-chosen submodule: the submodule $M' = (x_1, \ldots, x_k)$.

$$\begin{tikzcd} M' \cap N \arrow[r] \arrow[d] & N \arrow[d] \\ 0 \arrow[r] & N/(M' \cap N). \end{tikzcd}$$

First, we know that M/M' is generated by the image of x_{k+1}, so any submodule of M/M' is finitely generated. In particular, the image of N under the projection map $M \to M/M'$ is finitely generated. Choose lifts of these generators—we call them $y_1, \ldots, y_n$. They are element of M.

On the other hand, $M' \cap N$ is a submodule of M', hence finitely generated by the inductive hypothesis. Call the generators here $z_1, \ldots, z_p$.

Clearly, N is generated by the (finite) collection $y_1, \ldots, y_n, z_1, \ldots, z_p$—any element of N can be written as a linear combination of an element in the coset $y + (M' \cap N)$, where y is a lift of some element in $N/(M' \cap N)$. $\square$

Corollary 10.4. If A is a finitely generated abelian group, so is any subgroup.

Though this is completely unnecessary, we also see: If V is any finite-dimensional vector space over k, any subspace is also finite-dimensional. This is far more obvious a statement because we have bases, as opposed to just spanning sets.

10.1. Localization. I claimed last time that rings correspond to spaces—in one direction, given a space, you take the ring to be the collection of functions on that space.

And the most common way we create new spaces is by writing equations: That is, finding zero sets to equations.

So if we have a function (or collection of functions), we can look for the set of points on which the function(s) vanish; this is a new subspace we can play with.

But sometimes, we want to be clever.

Example 10.5. Recall that $GL_n(\mathbb{C})$ is the set of invertible $n \times n$ matrices. In this way, you probably think of it as a subset of $\mathbb{C}^{n \times n}$.

How do you define it as the zero set of some polynomial function?

The idea is that $\det \neq 0$ is not an equation; it's an inequality. So it's not immediately obvious how to define *complements* to an equation, like the complement of $\det = 0$.

But here's a trick: If det is to be nonzero, then there will always exist a (unique) number z for which $z \det = 1$.

Thus, consider the space

$$\mathbb{C}^{n^2+1} = \{((a_{ij}), z)\}.$$

There is a polynomial function on this space called

$$((a_{ij}), z) \mapsto z \det(a_{ij}) - 1.$$

This vanishes exactly along those pairs where (a_{ij}) is an invertible matrices, and z is the unique inverse to $\det(a_{ij})$.

The idea is to go "one dimension higher."

In terms of algebra, what is this doing? We are taking a function $f = \det$, and "formally adjoining a new variable" called z, then imposing the relation $zf = 1$.

In general, this process is called localization.

The philosophy I wanted to illustrate above is that it is possible not just to encode "zero sets" of functions, but also complements of such things.

Heuristically, suppose you want to take the complement of some set $Z \subset X$, and Z is defined by the vanishing of some functions $\{f_\alpha\}$.

Suppose now that g and h are two functions which *only* vanish along Z. If we multiply them together, we again get a function which *only* vanishes along Z. Moreover, as far as the complement of Z is concerned, g and h never vanish, so they admit multiplicative inverses on the complement of Z.

What this says is that the ring of functions of the complement of Z should look like a ring where we begin with R (the functions on X), and then we "invert" the functions which (if they vanish at all) only vanish on Z (or some subset thereof). The previous paragraph saw that such functions are closed under multiplication.

Definition 10.6. Let R be a ring. $U \subset R$ is called a *multiplicatively closed subset* if $g, h \in U \implies gh \in U$.

Example 10.7. Let $R = \mathbb{C}[t]$; this is the ring of functions on $\mathbb{C}$. What if we want to remove the origin? The only functions that vanish at the origin are those that are divisible by the polynomial t. In particular, t is a function which only vanishes at the origin, so let's invert t.

On the other hand, the prescription above says to add a formal variable and impose a relation. What we get is an isomorphism of rings

$$\mathbb{C}[t, t^{-1}] \cong \mathbb{C}[t, z]/(tz - 1).$$

In fact, if you like topology or geometry, you can go home and check that the spaces given by $\mathbb{C} \setminus \{0\}$ and $\{(t, z), tz - 1 = 0\} \subset \mathbb{C}^2$ are isomorphic.

Example 10.8. Let $R = \mathbb{Z}/15\mathbb{Z}$. Let $U = \{3, 6, 9, 12\}$. What is $U^{-1}R$? (See below.)

There is another process which looks like inverting certain ring elements, but not obviously so. It is the process of "only caring about what happens near a given subset."

Specifically, let's say that you only care about how a function behaves near some $Y \subset X$. If Z is a subset of X which never intersects Y (and is bounded away from Y), then as far as Y is concerned, the value of some function along Z is unimportant. In particular, if a function never vanishes near Y, but only vanishes along Z, we might as well invert it. We can do this over all Z which are bounded away from Y, and we get some new, gigantic ring of functions which has inverted every function that never vanishes near Y. In particular, two functions on X will be identified if they agree on some neighborhood near Y. In geometry, this is often called the "germ" of functions near Y.

As you may know from complex analysis, it is very hard for functions to be equal on some open subset without being equal entirely. Likewise, since complex polynomials are holomorphic functions, these "germs" in algebraic geometry have a very different feel from germs in topology, or for general smooth manifolds.

Example 10.9. Fix a prime p. Let $R = \mathbb{Z}$ and U be all numbers coprime to p.

Example 10.10. Fix an irreducible polynomial $f \in k[t]$, with k a field. Take U to be the set of all polynomials that are not divisible by f.

In the above cases, we are inverting by so many elements that we don't expect these new rings to be finitely generated over the old ones. So it's not so obvious that they'd be Noetherian. On the other hand, geometrically, it seems like "zooming into only care about what happens near Y" shouldn't destroy the finite-dimensionality of what we're studying.

Let's get concrete.

Definition 10.11. Let R be a ring, and $U \subset R$ a multiplicatively closed subset. Then $U^{-1}R$ is a new ring defined as follows:

(1) An element is an equivalence class of the pair (x, u), often written x/u, where $x \in R$ and $u \in U$. The relation is defined by

$$x/u \sim x'/u' \iff \text{there exists some } u_0 \in U \text{ so that } u_0(xu' - x'u) = 0.$$

(2) Addition is defined as though (x, u) were the fraction x/u:

$$(x/u) + (x'/u') = (xu' + x'u)/uu'.$$

(3) Multiplication is defined again as though (x, u) were a fraction:

$$(x/u)(x'/u') = (xx')/(uu').$$

Sometimes, one also writes $R[U^{-1}]$ for $U^{-1}[R]$. It is called the localization of R with respect to U.

DEFINITION 10.12. More generally, if M is an R-module, we can define a module $U^{-1}M$, which is a module over $U^{-1}R$, with the same formulas for addition as above (with $x \in M, u \in U$) and the obvious scaling action. Sometimes, one writes $M[U^{-1}]$ for $U^{-1}M$.

Note there is a natural ring homomorphism $\phi : R \to R[U^{-1}]$, given by sending $x \mapsto x/1$.

PROPOSITION 10.13. For any ideal $I \subset R[U^{-1}]$, we have that

$$I = \phi^{-1}(I)R[U^{-1}].$$

Note that we are using the ring map $R \to U^{-1}R$ to have $\phi^{-1}(I)$ act on $U^{-1}R$.

PROOF. The lefthand side is contained in the righthand side for obvious reasons—an ideal $J \subset R$ acts on $R[U^{-1}]$ via the homomorphism ϕ, so in this case, we have $\phi(\phi^{-1}(I))R[U^{-1}] = IR[U^{-1}] = I$, as I is an ideal.

On the other hand, if r/u is in I, then $r/1 = (r/u)u$ is in I because I is an ideal; hence r is in $\phi^{-1}(I)$. $\square$

COROLLARY 10.14. The assignment $I \mapsto \phi^{-1}(I)$, from ideals of $R[U^{-1}]$ to ideals of R, is an injection.

Note also that for any ring homomorphism $\phi : R \to S$, ϕ^{-1} preserves inclusions of sets.

COROLLARY 10.15. If R is Noetherian, so is $R[U^{-1}]$.

Importantly, this corollary is true even when $R[U^{-1}]$ may not even be a finitely generated algebra over R.

REMARK 10.16. One can do better: It turns out that when restricted to prime ideals, the assignment $I \mapsto \phi^{-1}(I)$ becomes a bijection between prime ideals of the localization, and the prime ideals of R that do not intersect U.