Lecture 9: Noetherian rings and modules

Today we begin with some commutative algebra.

I'm geometrically minded, so a lot of motivation will be from geometry.

The idea of algebraic geometry is to turn geometric questions into algebraic ones, and vice versa. To do this, we want a dictionary. The dictionary goes like this, but the following is very rough, and literally incorrect. We'll see details later.

Algebra	Geometry	A pathway
A ring R	A space $X = \operatorname{Spec} R$	Space $X \mapsto$ Functions on X.
$\mathbb{C}$	$X =$ one point.	
$\mathbb{C}[x]$	The space $\mathbb{C}$	
$\mathbb{C}[x_1, \ldots, x_n]$	The space $\mathbb{C}^n$	
$R[x]$	(The space associated to R)$\times\mathbb{C}$	
Ideal $I \subset R$	A subspace $V(I) \subset \operatorname{Spec} R$	$V(I)$=points on which all $f \in I$ evaluate to 0.
R/I	Functions on $V(I)$	
Ring map $R \to S$	Maps $\operatorname{Spec} S \to \operatorname{Spec} R$	Functions on R pull back to functions on S
$R \to R/I$	Inclusion $V(I) \to \operatorname{Spec} R$	
R is Noetherian	$\operatorname{Spec} R$ is "finite-dimensional."	

Being "Noetherian" is like being finite-dimensional.

Remark 9.1. The zero ring corresponds to the empty set.

Remark 9.2. Choosing a "base ring" determined what you think of as a point. Above, we chose $\mathbb{C}$ as a base ring, so $\mathbb{C}$ is functions on a point.

From now on, every ring R is commutative and has unit.

Proposition 9.3. Let R be a ring, and M a module. The following are equivalent:

(1) Every submodule of M is finitely generated.
(2) Any *strictly* ascending sequence of submodules in M

$$M_0 \subset M_1 \subset \ldots$$

must terminate (after a finite number of steps).

PROOF. (1) $\implies$ (2). If every submodule of M is finitely generated, so is the union $\cup_i M_i$. Letting $x_1, \dots, x_k$ be a set of generators, by definition of union, there exists a finite l so that each x_i is contained in M_l. This means $\cup_i M_i = M_l$, so the sequence terminates with a module equal to M_l.

(2) $\implies$ (1). If M' is any submodule of M, make a sequence of submodules of M' as follows: Choose $x_0 \in M'$. Then if the R-module generated by x_0, written (x_0), does not equal M', choose x_1 not in (x_0). Inductively, if $(x_0, \dots, x_i) \neq M'$, choose x_{i+1} to be an element in $M' \setminus (x_0, \dots, x_i)$. This creates a sequence of submodules

$$(x_0) \subset (x_0, x_1) \subset \dots$$

which must terminate. Hence at some finite l, we have that

$$(x_0, \dots, x_l) = M'$$

and M' is finitely generated. □

DEFINITION 9.4. Let R be a ring. An ideal $I \subset R$ is a submodule of R. Concretely, this means

(1) I is an abelian subgroup (under addition), and
(2) For any $x \in I, r \in R$, we have that $rx \in I$.

This implies that any finite linear combination

$$\sum r_i x_i$$

is also in I.

DEFINITION 9.5. A ring R is called Noetherian if it satisfies either of the conditions in the proposition.

An R-module M (regardless of whether R is Noetherian) is called Noetherian if M satisfies either of the conditions in the proposition.

COROLLARY 9.6. If R is Noetherian, so is the image of any ring homomorphism out of R.

PROOF. If $\phi : R \to S$ is a surjection, then $S \cong R/\ker(I)$. Since ideals of S are in bijection with ideals of R containing $\ker(I)$, the ascending chain condition holds. □

EXAMPLE 9.7. $\mathbb{Z}$ is Noetherian because all its ideals are finitely generated (in fact, all its ideals are of the form (n) for some $n \in \mathbb{Z}$; hence generated by a single element).

Any field k is Noetherian because a field only has two ideals: 0 and k.

For any field k, $k[t]$ is Noetherian—by the Euclidean algorithm, any ideal is generated by a single element.

The following is called the Hilbert basis theorem. It jives with the following intuition: If X is a finite-dimensional space, it's still finite-dimensional if you take its direct product with a one-dimensional line. In algebra, if you have a ring R, then the way to attach one free dimension to R is to attach a free variable to construct the polynomial ring R.

THEOREM 9.8 (Hilbert basis theorem). Let R be Noetherian. Then $R[x]$ is Noetherian.

PROOF. All we have is that polynomials have degree. We use the notion of degree to show any ideal is finitely generated.

Let $I \subset R[x]$ be an ideal. We construct a sequence of subideals as follows:

Choose $f_0 \in I$ to be a minimal degree polynomial in I. (It may not be unique; two f_0 may even generate a different subideal.)

Inductively: If $(f_0, \ldots, f_i) \neq i$, choose an element f_{i+1} of minimal degree that is in the complement $I \setminus (f_0, \ldots, f_i)$.

Let a_i be the leading coefficient of f_i (i.e., the highest degree coefficient). Then $(a_0, a_1, \ldots)$ is an ideal of R. By hypothesis, this ideal is finitely generated, so there is some finite collection $(a_0, \ldots, a_m)$ which generates it.

Here comes a contradiction about the existence of f_{m+1}: By the choice of $a_0, \ldots, a_m$, we know that the leading coefficient a_{m+1} can be generated as an R-linear combination of $a_0, \ldots, a_m$:

$$a_{m+1} = \sum_{i=0}^{m} u_i a_i.$$

So if $\deg f_{m+1} = N$, consider the linear combination

$$g = \sum_{i=0}^{m} u_I x^{N - \deg f_i} f_i.$$

This is a new polynomial whose leading coefficient is a_{m+1}, and has degree equal to the degree of f_{m+1}. Note that it is in the ideal generated by $(f_0, \ldots, f_m)$, so the difference

$$f_{m+1} - g$$

better not be in the ideal $(f_0, \ldots, f_m)$. This contradicts the existence of f_{m+1}, because this difference is a lower-degree polynomial than f_{m+1}. $\square$