

Lecture 11: Localizations are Noetherian; prime and maximal ideals

Here is an exercise:

EXERCISE 11.1. Note that the map $x \mapsto x/1$ is a ring homomorphism from R to $U^{-1}R$. Show that the map sending ideals $I \subset U^{-1}(R)$ to their preimages $\tilde{I} \subset R$ is an injection.

COROLLARY 11.2. If R is Noetherian, so is $U^{-1}R$.

PROOF OF COROLLARY. Because the map sending I to $\tilde{I}$ is an injection, any ascending chain in $U^{-1}(R)$ must terminate given its preimage terminates in R . $\square$

PROOF. Note that $I = \tilde{I}U^{-1}R$.

Note $\tilde{I}$ consists of those elements of I expressible in the form $x/1$. Then any element of I given by x'/r' is written $x'/1 \cdot 1/r'$, hence $I \subset \tilde{I}U^{-1}R$. We conclude $I \subset \tilde{I}U^{-1}R$.

The other inclusion is obvious. $\square$

The reason this is new: $U^{-1}R$ need not be finitely generated as an R -algebra, so the Hilbert basis theorem may not apply. Geometrically: If localization is like looking at complements, certainly a complement of a subspace of a finite-dimensional thing is still finite-dimensional.

11.2. More on the geometry-algebra dictionary: Empty spaces and points. The correspondence between rings and spaces reverses arrows: If you a map of spaces $X \rightarrow Y$, one has a map of rings $\mathcal{O}_Y \rightarrow \mathcal{O}_X$ by pulling back functions.

The empty set is initial: Given any other set X , there is a unique function $\emptyset \rightarrow X$. This characterizes the empty set among all spaces.

Likewise, the zero ring is final: Given any other ring R , there is a unique ring map $R \rightarrow 0$. This characterizes the zero ring among all rings, and this zero ring is the “ring of functions” of the empty space.

The corresponding becomes super interesting when we consider points. There are at least two different ways to characterize the space/set called “a single point.”

(1) It is terminal in spaces; given any other X , there is a unique map $X \rightarrow pt$.

(2) The power set consists of two elements: The empty set, and the point. That is, there are exactly two subspaces of a point.

Now, property (1) is enjoyed by the ring $\mathbb{Z}$ when arrows are reversed. Given any commutative unital ring R , there is a unique ring map $\mathbb{Z} \rightarrow R$ by sending $1 \mapsto 1$. So $\mathbb{Z}$ acts like a point.

On the other hand, property (2) is enjoyed precisely by every field. (A ring has exactly two ideals if and only if a ring is a field.) So it looks like there are many, many points—one for every field.

REMARK 11.3. We may later see that fields are more like places where functions are allowed to take values; any point must have functions taking values somewhere, and these “somewheres” will always be a field.

11.3. Prime and maximal ideals. To make the geometry-algebra dictionary more precise, we go over some notions.

PROPOSITION 11.4. Fix an ideal $I \subset R$ which is proper (i.e., so $I \neq R$). The following are equivalent:

- (1) R/I is an integral domain. That is, it has no zero divisors: If $\underline{xy} = 0$, then either $\underline{x}$ or $\underline{y}$ are equal to zero in R/I .
- (2) If the product xy is in I , then either x or y is in I . (Maybe both.)

PROOF. (1) $\implies$ (2): We let $\underline{x}$ denote the image of $x \in R$ in the quotient R/I . If $xy \in I$, then $\underline{xy} = \underline{0}$. By (1), this means either $\underline{x}$ or $\underline{y}$ is equal to $\underline{0}$, meaning either x or y is in the ideal I (because the kernel of the quotient map is I).

(2) $\implies$ (1): Similar. □

DEFINITION 11.5. A proper ideal $I \subset R$ satisfying either (hence both) of the above conditions is called a *prime ideal*.

EXAMPLE 11.6. Let $R = \mathbb{Z}$. Then for any prime p , $(p) = p\mathbb{Z}$ is a prime ideal since $\mathbb{Z}/p\mathbb{Z}$ is a field (hence has no zero divisors).

EXAMPLE 11.7 (Non-example). If $R = \mathbb{C}[x, y]$, then the ideal $I = (xy)$ is not a prime ideal.

PROPOSITION 11.8. Fix an ideal $I \subset R$ which is proper (i.e., so $I \neq R$). The following are equivalent:

- (1) R/I is a field.
- (2) The only ideal containing I is R itself.

PROOF. (1) $\implies$ (2): There is a bijection between ideals of R/I and the ideals of R containing I . Since a field has only one non-zero ideal, (2) follows. (2) $\implies$ (1): Similar. Any non-zero ring (we know R/I is non-zero because $R \neq I$) has at least two ideals: 0 and itself. It has two exactly when it is a field. □

DEFINITION 11.9. A proper ideal $I \subset R$ satisfying either (hence both) of the above conditions is called a *maximal ideal*.

EXAMPLE 11.10. Let $R = \mathbb{Z}$. Then for any prime p , $(p) = p\mathbb{Z}$ is a prime ideal since $\mathbb{Z}/p\mathbb{Z}$ is a field (hence has no zero divisors).