Lecture 12: Zariski topologies.

Exercise 12.1. (1) Show that if I is maximal, it is prime.
(2) Exhibit an ideal of $\mathbb{Z}$ which is prime but not maximal.
(3) Show that if $\mathfrak{p} \subset S$ is prime, then for any ring homomorphism $f : R \to S$, the pre-image $f^{-1}(\mathfrak{p})$ is prime.

Proof. (1) If I is maximal, R/I is a field, hence has no zero divisors.
(2) The ideal (0) is prime but not maximal.
(3) Consider the composition $R \to S \to S/\mathfrak{p}$. This is a ring homomorphism with kernel $f^{-1}(\mathfrak{P})$. By the first isomorphism theorem, $R/(f^{-1}(\mathfrak{p}))$ is isomorphic (as a ring) to its image in $S/\mathfrak{p}$, but the latter has no zero divisors, so neither does this image.

$\square$

12.4. The Zariski Topologies.

Definition 12.2. Let R be a ring. Then $Spec(R)$ is the set of prime ideals of R.

Given a subset $E \subset R$, let $V(E) \subset Spec(R)$ denote the set of all prime ideals which contain E.

Theorem 12.3 (Homework). The sets of the form $V(E)$ form a collection of closed subsets for a topology on $Spec(R)$.

Definition 12.4. This is called the *Zariski topology* of $Spec(R)$.

Example 12.5. $Spec(\mathbb{Z})$ is in bijection with $\{0\} \cup \{primes\}$. Every prime number defines a prime ideal $p\mathbb{Z}$, and because $p\mathbb{Z} = V(\{p\})$, these points are also closed subsets. Likewise, something like $V(2,3) = \{(2),(3)\}$ is a closed subset—it is the union of two closed points.

Surprisingly, the point $(0) \in Spec(\mathbb{Z})$ is not closed. Its closure is the entire space $Spec(\mathbb{Z})$. This (0) is an example of a *generic point* of $Spec(\mathbb{Z})$.

Finally, $V(\mathbb{Z})$ is the empty set.

Definition 12.6. Let $MaxSpec(R) \subset Spec(R)$ denote the collection of maximal ideals. We endow it with the subspace topology.

Remark 12.7. The subspace topology for a subset $Y \subset X$ is defined as follows: A subset $K_Y \subset Y$ is closed iff $K_Y = K \cap Y$ for some $K \subset X$ closed in X.

The following works for any algebraic subset of $\mathbb{C}^n$, but we define it only for $\mathbb{C}^n$. Take the subspace topology if you like, and this defines the Zariski topology for any algebraic subset.

Definition 12.8 (Another Zariski topology). Fix an ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$. We let $V(I) \subset \mathbb{C}^n$ denote the set of those $(z_1, \ldots, z_n)$ such that every element of I vanishes on $(z_1, \ldots, z_n)$.

Call a subset $Y \subset \mathbb{C}^n$ *closed* if and only if there exists an ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ such that $Y = V(I)$. (I.e., if Y is an algebraic subset.)

Claim: This defines a topology on $\mathbb{C}^n$. We won't give a proof yet; it's analogous to your homework.

Remark 12.9. This is a far smaller topology than the usual one on $\mathbb{C}^n$; for instance, the only closed balls in this topology are either empty, radius-zero, or radius-infinity.

Theorem 12.10 (Nullstellensatz, second consequence). The function

$$\mathbb{C}^n \to MaxSpec(\mathbb{C}[x_1, \ldots, x_n]), \qquad (z_1, \ldots, z_n) \mapsto (x_1 - z_1, \ldots, z_n - z_n)$$

is a homeomorphism.

So while $Spec(\mathbb{C}[x_1, \ldots, x_n])$ may still be interesting, the two Zariski topologies on $\mathbb{C}^n \cong MaxSpec(\mathbb{C}[x_1, \ldots, x_n])$ agree.

So this lecture was another motivation for the Nullstellensatz, whatever it is.