Lecture 14: Categories and Functors

We review some basic category theory.

14.1. Categories. Category theory was invented to be able to relate different languages of math together. Vaguely, here's a table:

TABLE 1. default

The objects we care about	The functions we care about
Sets	Functions
Groups	Group homomorphisms
Rings	Ring homomorphisms
Topological spaces	Continuous functions
Smooth manifolds	Smooth maps

So we have *objects*, and *ways to compare objects;* i.e., homomorphisms.

DEFINITION 14.1. A category $\mathcal{C}$ is the data of:

(1) A collection $\mathrm{Ob}\,\mathcal{C}$; we call an element $X \in \mathrm{Ob}\,\mathcal{C}$ an *object* of $\mathcal{C}$. In the category of groups, every group is an object.[1]

(2) For every pair of objects $X, Y \in \mathrm{Ob}\,\mathcal{C}$, a set $\hom(X, Y)$, called the set of morphisms from X to Y. In the category of groups, $\hom(X, Y)$ is the set of group homomorphisms from X to Y.

(3) For every triple of objects $X, Y, Z \in \mathrm{Ob}\,\mathcal{C}$, a function

$$\circ : \hom(Y, Z) \times \hom(X, Y) \to \hom(X, Z), \qquad (f_{YZ}, f_{XY}) \mapsto f_{YZ} \circ f_{XY}$$

called composition. In the category of groups, this is usual composition of homomorphisms.

These data must satisfy the following conditions:

[1] I am using the word "collection" to avoid set-theoretic issues; for instance, there is no set of sets. We ignore such issues altogether.

(4) For every object $X \in \mathrm{Ob}\,\mathcal{C}$, an element $\mathrm{id}_X \in \hom(X,X)$ called the *identity of* X; it must satisfy the property that for all $Y \in \mathrm{Ob}\,\mathcal{C}$ and all $f_{XY} \in \hom(X,Y)$, $f_{YX} \in \hom(Y,X)$, we have

$$\mathrm{id}_X \circ f_{YX} = f_{YX}, \qquad f_{XY} \circ \mathrm{id}_X = f_{XY}.$$

In the category of groups, id_X is the identity automorphism from X to itself.

(5) Associativity: For every quadruple of objects W, X, Y, Z, and for every triple of morphisms $f_{ab} \in \hom(a,b)$, an equality

$$(f_{YZ} \circ f_{XY}) \circ f_{WX} f_{YZ} \circ (f_{XY} \circ f_{WX}).$$

Composition of group homomorphisms is associative.

Example 14.2. The blue above shows that there's a category of groups with objects and morphisms as you'd expect from the table. Likewise there is a category of rings, of topological spaces, of smooth manifolds, of sets.

Example 14.3. Another example is as follows: Fix a category $\mathcal{C}$ with only one object, so $\mathrm{Ob}\,\mathcal{C} = \{*\}$. Then there is a single morphism set $A := \hom(*,*)$ to specify, along with a composition map $A \times A \to A$.

These data must admit a unit $1 \in A$, and the composition $A \times A \to A$ must be associative. That is, a category with only one object is the same data as that of an associative monoid (i.e., a group that may not admit inverses).

More generally, for any category $\mathcal{C}$, any object $X \in \mathrm{Ob}\,\mathcal{C}$ determines an associative monoid called $\hom(X,X)$.

The term "isomorphism" has a meaning in this general context:

Definition 14.4. Let $\mathcal{C}$ be a category and choose a morphism $f \in \hom_{\mathcal{C}}(X,Y)$. We say that f is an *isomorphism* if there exists $g \in \hom_{\mathcal{C}}(Y,X)$ such that

$$fg = \mathrm{id}_Y, \qquad gf = \mathrm{id}_X.$$

14.2. Functors. Even better, if we now decide that we care about categories (at the very least, we have interesting examples), we should ask about what the "functions" or "relations" we care about are:

I've appended the table: The notion of a "morphism of categories" is a functor. It's a key new word.

TABLE 2. default

The objects we care about	The functions we care about
Sets	Functions
Groups	Group homomorphisms
Rings	Ring homomorphisms
Topological spaces	Continuous functions
Smooth manifolds	Smooth maps
Categories	Functors

DEFINITION 14.5. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Then a *functor* $F : \mathcal{C} \to \mathcal{D}$ is the data of[2]

(1) A function $F : \mathrm{Ob}\,\mathcal{C} \to \mathcal{D}$ For example, if $\mathcal{C}$ is the category of groups and $\mathcal{D}$ is the category of associative rings, we could have a function that sends any group G to the ring $\mathbb{C}G$.
(2) For every pair of objects $X, Y \in \mathrm{Ob}\,\mathcal{C}$, a function
$$F : \hom_{\mathcal{C}}(X, Y) \to \hom_{\mathcal{D}}(FX, FY).$$
For example, for any group homomorphism $G \to H$, we assign the obvious ring homomorphism $\mathbb{C}G \to \mathbb{C}H$..
And this data must satisfy the following:
(3) F respects units, so for any X, $F(\mathrm{id}_X) = \mathrm{id}_{FX}$,
(4) F respects compositions, so
$$F(f_{YZ} \circ f_{XY}) = F(f_{YZ}) \circ F(f_{XY}).$$

EXAMPLE 14.6. The blue example above shows that the assignment $G \mapsto \mathbb{C}G$, along with the obvious assignment on homomorphisms, is a functor from Groups to Rings.

EXAMPLE 14.7. Another example: If $\mathcal{C}$ and $\mathcal{D}$ each has only one object, then a functor is just a map of associative monoids.

14.3. More examples of categories.

EXAMPLE 14.8. The category of categories, denoted Cat. $\mathrm{Ob}\,\mathsf{Cat}$ consists of all categories (ignoring set-theoretical issues; this is actually delicate). Given two categories $\mathcal{C}$ and $\mathcal{D}$, we let $\hom_{\mathsf{Cat}}(\mathcal{C}, \mathcal{D}) = \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ denote the set of functors from $\mathcal{C}$ to $\mathcal{D}$.

[2] Confusingly, both the functor F and the functions in (1) and (2) are denoted F. This is fairly common.

EXAMPLE 14.9 (Sets). Let S be a set. Then one can define a category $\mathcal{C}$ where $\operatorname{Ob}\mathcal{C} = S$, $\hom(x,x) = *$ is a point for every $x \in S$, and $\hom(x,y) = \emptyset$ for any $x \neq y$.

In this way, one can think of a set as a category where there are no morphisms between any objects except the identity morphism from an object to itself.

Notationally, if S is a set, we will also let S denote the category described above. Believe it or not, it will be meaningful to think about functors from S to an arbitrary category $\mathcal{D}$ when we later discuss limits and colimits.

EXAMPLE 14.10 (Posets). Recall that a *partially ordered set* $(P, \leq)$ is a set P together with a partial order relation $\leq$. This is a relation which satisfies:

(1) $x \leq x$ for all $x \in P$.
(2) If $x \leq y$ and $y \leq x$, then $x = y$.
(3) If $x \leq y$ and $y \leq z$, then $x \leq z$. (Transitivity.)

Then any poset P defines a category $\mathcal{C}$ as follows: $\operatorname{Ob}\mathcal{C} = P$, and $\hom(x,y)$ is empty unless $x \leq y$. If $x \leq y$, then $\hom(x,y) = *$ is a singleton.

Believe it or not, the example of the posets

$$(\mathbb{Z}_{\geq 0}, \leq) \qquad \text{and} \qquad (\mathbb{Z}_{\leq 0}, \leq)$$

will both be important when we consider limits and colimits later.

Notationally, we will write $(P, \leq)$ or even just P for the category given by a poset.

EXAMPLE 14.11 (Opposites). Given a category $\mathcal{C}$, one can construct another category $\mathcal{C}^{\mathrm{op}}$. We define it by $\operatorname{Ob}\mathcal{C}^{\mathrm{op}} = \operatorname{Ob}\mathcal{C}$, and for $x, y \in \mathcal{C}^{\mathrm{op}}$, we set

$$\hom_{\mathcal{C}^{\mathrm{op}}}(x,y) := \hom_{\mathcal{C}}(y,x).$$

The composition maps are the obvious ones:

$$\begin{aligned}
\hom_{\mathcal{C}^{\mathrm{op}}}(y,z) \times \hom_{\mathcal{C}^{\mathrm{op}}}(x,y) &= \hom_{\mathcal{C}}(z,y) \times \hom_{\mathcal{C}}(y,x) \\
&\cong \hom_{\mathcal{C}}(y,x) \times \hom_{\mathcal{C}}(z,y) \\
&\to \hom_{\mathcal{C}}(z,x) \\
&= \hom_{\mathcal{C}^{\mathrm{op}}}(x,z).
\end{aligned}$$

You should check this is associative.

Example 14.12 (Products). Given two categories $\mathcal{C}$ and $\mathcal{D}$, their product category $\mathcal{C}\times\mathcal{D}$ is defined as follows: $\mathrm{Ob}(\mathcal{C}\times\mathcal{D}) = \mathrm{Ob}\,\mathcal{C}\times\mathrm{Ob}\,\mathcal{D}$, and

$$\hom_{\mathcal{C}\times\mathcal{D}}((x,x'),(y,y')) := \hom_{\mathcal{C}}(x,y)\times\hom_{\mathcal{D}}(x',y')$$

with the obvious composition maps.

Example 14.13 (S-algebras). Fix a ring S. We can form a category $S\mathsf{Alg}$ whose objects are S-algebras (i.e., a ring R with a ring map $S\to R$) and whose morphisms are maps of S-algebras (i.e., ring maps $R\to R'$ which are a map of S-modules).

14.4. More examples of functors.

Example 14.14. We already gave the example of a functor $\mathsf{Groups}\to\mathsf{Rings}$ which sends a group G to its group ring $\mathbb{C}G$, and which sends a group homomorphism to its induced ring map. Note there is such a functor for any choice of base ring k (i.e., k need not equal $\mathbb{C}$).

Example 14.15. Here is another example: Given a ring S, we have its category of S-modules. Moreover, given a ring map $S\to T$, we can turn any S-module M into an R-module by taking $T\otimes_S M$. This also turns any S-module map $f: M\to N$ into a T-module map $\mathrm{id}_T\otimes f$, and defines a functor

$$\mathsf{Rings}\to\mathsf{Cat}$$

from the category of rings to the category of categories.

Example 14.16 (Forget). Given any group, one can forget its group structure and consider it just as set. Since any group homomorphism is in particular a function, we have a functor

$$\mathsf{Groups}\to\mathsf{Sets}.$$

Likewise, we have "forgetful" functors

$$\mathsf{Rings}\to\mathsf{Sets},\qquad \mathsf{Spaces}\to\mathsf{Sets},$$

et cetera.

Example 14.17 (Free). Given a set, we can consider the free group generated by that set. And any function $X\to Y$ induces a group homomorphism $Free(X)\to Free(Y)$. So we also have a functor

$$\mathsf{Sets}\to\mathsf{Groups}.$$

14.5. Natural Transformations and equivalences of categories. Here is an astounding lesson in life: Isomorphisms are not always the correct notion of equivalence.

Perhaps the better lesson to be learned is: Assignments should be allowed to be inverses "up to" a reasonable ambiguity.

DEFINITION 14.18. Let F and G be functors from $\mathcal{C}$ to $\mathcal{D}$. A *natural transformation* from F to G is a choice $\eta_X \in \hom_{\mathcal{D}}(FX, GX)$ for every $X \in \operatorname{Ob}\mathcal{C}$ such that, for any morphism $f : X \to X'$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(X') \arrow[d, "\eta_{X'}"] \\ G(X) \arrow[r, "G(f)"] & G(X') \end{tikzcd}$$

A natural transformation is called a *natural isomorphism* if every η_X is an isomorphism.

Here's the idea behind the following definition: When should two categories $\mathcal{C}$ and $\mathcal{D}$ be considered "the same?" Well, certainly there should be a correspondence between $\operatorname{Ob}\mathcal{C}$ and $\operatorname{Ob}\mathcal{D}$, and whatever this correspondence is, it should also include some bijection between the appropriate morphism spaces (i.e., between the hom sets). This looks like one wants the notion of an "isomorphism" for functors; a functor $F : \mathcal{C} \to \mathcal{D}$ which is a bijection on $\operatorname{Ob}\mathcal{C} \cong \operatorname{Ob}\mathcal{D}$ and a bijection $\hom(X, Y) \cong \hom_{\mathcal{D}}(FX, FY)$.

But here's another insight: Let's say we have a functor which doesn't hit every object of $\mathcal{D}$, but all objects of $\mathcal{D}$ are hit *up to isomorphism.* In math, we only case about individual objects of $\mathcal{C}$ or $\mathcal{D}$ up to isomorphism, so shouldn't this be "enough" to consider F as some sort of equivalence? After all, all the algebraic information is already being captured.

DEFINITION 14.19. A functor $F : \mathcal{C} \to \mathcal{D}$ is called an *isomorphism* if the maps $\operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{D}$ and $\hom(X, Y) \to \hom(FX, FY)$ are all bijections.

A functor $\mathcal{C} \to \mathcal{D}$ is called an *equivalence of categories* if

(1) for any $D \in \operatorname{Ob} cD$, there exists an object $C \in \mathcal{C}$ such that $F(C)$ is isomorphic to D, (i.e., F is *essentially surjective*) and
(2) For any $X, Y \in \operatorname{Ob}\mathcal{C}$, the map $\hom(X, Y) \to \hom_{\mathcal{D}}(FX, FY)$ is a bijection (i.e., F is *fully faithful*).

Remark 14.20. For more "combinatorial" applications of category theory, the notion of isomorphism of categories can be very useful and important.

For more "algebraic" or "structural" applications of category theory, the notion of equivalence is far more useful.

Exercise 14.21. Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of categories. Show that there is a full subcategory $\mathcal{C}' \subset \mathcal{C}$ such that if F' is the restriction of F to $\mathcal{C}'$, F' is an isomorphism onto $F(\mathcal{C})$.

Exercise 14.22. Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of categories. Show that there exists a functor $G : \mathcal{D} \to \mathcal{F}$ and natural isomorphisms $F \circ G \to \mathrm{id}_{\mathcal{D}}$, $G \circ F \to \mathrm{id}_{\mathcal{C}}$.

Prove the converse. This proves the sense in which F is like an "isomorphism of categories up to natural isomorphism".

14.6. More examples.

Example 14.23. Let $\mathcal{C}$ be a category. Then for any object $Y \in \mathrm{Ob}\,\mathcal{C}$, there is a functor

$$\mathcal{Y}_Y : \mathcal{C}^{\mathrm{op}} \to \mathsf{Sets}$$

which sends an object X to the set $\hom(X, Y)$.

Exercise: Prove this is a functor.

Example 14.24 (Yoneda embedding). In fact, since the above assignment works for any object of $\mathcal{C}$, one might expect it to form a functor as follows:

$$\mathcal{Y} : \mathcal{C} \to \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Sets}), \qquad Y \mapsto \mathcal{Y}_Y.$$

Indeed one can; a map $Y \to Y'$ induces a natural transformation $\mathcal{Y}_Y \to \mathcal{Y}_{Y'}$.

Exercise: Prove this is a fully faithful functor (that is, the map on morphism sets is a bijection).

Example 14.25. Let $\mathcal{C}$ be a category with one object, which we identify with a monoid M. What are the natural transformations of the identity functor?

Answer: The center of M.