

Lecture 13: Tensor products as fiber products.

We continue our dictionary to motivate new definitions.

DEFINITION 13.1. Fix three spaces X, Y, Z and functions $f : X \rightarrow Z, g : Y \rightarrow Z$. The *fiber product* is defined as

$$X \times_Z Y := \{(x, y) \text{ such that } f(x) = g(y)\}.$$

EXAMPLE 13.2. If X is a point and g is the inclusion of $z_0 \in Z$, then $X \times_Z Y \cong g^{-1}(z_0)$.

If X, Y are subsets of Z and f, g are the inclusion maps, then $X \times_Z Y \cong X \cap Y$.

REMARK 13.3. In the above remark, I use the bijection notation \cong rather than the equality notation $=$. This is because, literally, these sets are *not equal*. For example, $X \times_Z Y$ is always a subset of $X \times Y$, while $X \cap Y$ is a subset of Z in the last example.

REMARK 13.4. Also, note that the maps f and g are omitted from the notation $X \times_Z Y$; they are to be understood implicitly.

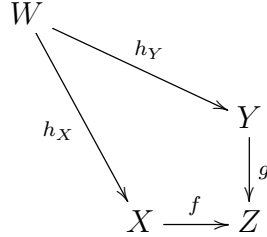
The fiber product satisfies a universal property. First, fix the projection map

$$p_X : X \times_Z Y \rightarrow X, \quad p_X(x, y) = x$$

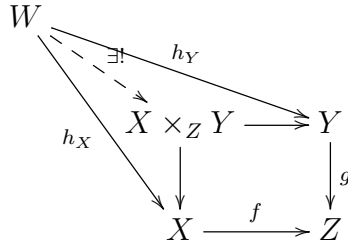
and likewise for p_Y .

THEOREM 13.5. The data $(X \times_Z Y, p_X, p_Y)$ satisfy the following universal property:

Given any space/set W together with functions $h_X : W \rightarrow X, h_Y : W \rightarrow Y$ making the diagram



commute, then there exists a unique function $j : W \rightarrow X \times_Z Y$ making the following commute:



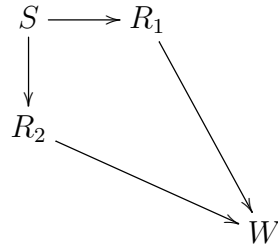
Fiber products are important to generalize because they capture things like intersections and pre-images (as the examples above show). Now let's use our geometry/algebra dictionary again: Remember that a function $f : X \rightarrow Y$ has its arrow direction reversed: $f^* : \mathcal{O}_Y \rightarrow \mathcal{O}_X$, where \mathcal{O}_\bullet is the ring of functions of \bullet .

THEOREM 13.6. Let S, R_1, R_2 be rings, and fix ring homomorphisms $S \rightarrow R_1, S \rightarrow R_2$. Then there exists a ring $R_1 \otimes_S R_2$, together with ring maps

$$R_1 \rightarrow R_1 \otimes_S R_2, \quad R_2 \rightarrow R_1 \otimes_S R_2,$$

such that the following universal property is satisfied:

For any ring W equipped with maps $R_1 \rightarrow W, R_2 \rightarrow W$ making the following diagram commute:



there exists a unique ring map $j : R_1 \otimes_S R_2 \rightarrow W$ making the following diagram commute:

$$\begin{array}{ccc}
 S & \longrightarrow & R_1 \\
 \downarrow & & \downarrow \\
 R_2 & \longrightarrow & R_1 \otimes_S R_2 \\
 & \searrow & \nearrow \text{---} \\
 & & W
 \end{array}$$

REMARK 13.7. These diagrams are obtained by simply reversing the arrows in the previous diagrams about sets/spaces.

We will exhibit this ring in two steps. First, let's understand what it is as a module over S . We'll exhibit the multiplication later.

THEOREM 13.8. Let S be a ring and M_1, M_2 two S -modules. Then there exists a module $M_1 \otimes_S M_2$, together with an S -bilinear map $M_1 \times M_2 \rightarrow M_1 \otimes_S M_2$, such that the following universal property is satisfied:

Let N be any module equipped with a S -bilinear map $\phi : M_1 \times M_2 \rightarrow N$. Then there exists a unique S -linear map $M_1 \otimes_S M_2 \rightarrow N$ making the following diagram commute:

$$\begin{array}{ccc}
 M_1 \times M_2 & \xrightarrow{\phi} & N \\
 \downarrow & \searrow \exists! & \\
 M_1 \otimes_S M_2 & &
 \end{array}$$

PROOF. This will be quick given that we've seen this for vector spaces before. The construction is the same; it proceeds in two steps.

Step One: Let $S\langle M_1 \times M_2 \rangle$ be the free S -module on the set $M_1 \times M_2$. It has a set of generators in bijection with the set $M_1 \times M_2$, and we label a generator by a pair: e_{m_1, m_2} .

Step Two: Consider the S -submodule $B \subset S\langle M_1 \times M_2 \rangle$ generated by the following elements:

- (1) $se_{m_1, m_2} - e_{sm_1, m_2}$
- (2) $se_{m_1, m_2} - e_{m_1, sm_2}$
- (3) $e_{m_1 + m'_1, m_2} - e_{m_1, m_2} - e_{m'_1, m_2}$
- (4) $e_{m_1, m_2 + m'_2} - e_{m_1, m_2} - e_{m_1, m'_2}$.

We let $M_1 \otimes_S M_2$ be the quotient by B .

Now we have the following maps:

$$\begin{array}{ccc}
 M_1 \times M_2 & \xrightarrow{\phi} & N \\
 \downarrow & \nearrow \tilde{\phi} & \\
 S\langle M_1 \times M_2 \rangle & & \\
 \downarrow & \nearrow \exists! & \\
 M_1 \otimes_S M_2 & &
 \end{array}$$

The first downward map is a function; it includes the basis vectors. Then ϕ determines a linear map $\tilde{\phi}$ uniquely by extending S -linearly.

Finally, since ϕ is bilinear, every element of B is sent to $0 \in N$ by $\tilde{\phi}$. It follows that there is a unique linear map from the quotient to N making the diagram commute. \square

EXAMPLE 13.9. $\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/5\mathbb{Z} = 0$.

Why? Note that any bilinear map out of $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ has to be zero: This is because $\phi(1, a) = \phi(5 \cdot 2, a) = 5\phi(2, a) = \phi(2, 5a) = \phi(2, 0) = 0$. Likewise, $\phi(x, 1) = 0$ for any x . In particular, the identity morphism from the tensor product to itself must be zero.

REMARK 13.10. More generally, example 13.9 generalizes to show that $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$ tensor to the zero ring. There's a geometric interpretation: Inside $\text{Spec } \mathbb{Z}$, the points (p) and (q) are disjoint, so they have empty intersection. (The zero ring is the ring of functions of the empty set.)

13.5. The language of S -algebras. Oftentimes we have a base ring in mind. For example, $\mathbb{C}[x_1, \dots, x_n]$ seems to take \mathbb{C} as a starting point; likewise for the group ring $\mathbb{C}G$.

PROPOSITION 13.11. Fix two rings S and R . The following are equivalent:

- (1) The data of a ring map $S \rightarrow R$.
- (2) The data of an S -module structure on R , and the S -bilinearity of the ring multiplication $R \times R \rightarrow R$.

DEFINITION 13.12. A ring R equipped with either (hence both) of (1) or (2) is called an S -algebra.

PROOF. (1) \implies (2). Let $\phi : R \rightarrow S$ be the ring map. Consider the composition

$$S \times R \rightarrow R \times R \rightarrow R, \quad (s, r) \mapsto (\phi(s), r) \mapsto \phi(s) \cdot r.$$

This is an S -module structure on R . The multiplication map $m : R \times R \rightarrow R$ is obviously additive in each factor; by associativity and commutativity, we also have:

$$m(\phi(s)r, r') = \phi(s)rr' = r\phi(s)r' = m(r, \phi(s)r')$$

so it's S -bilinear.

(2) \implies (1). Consider the S -module structure $\mu : S \times R \rightarrow R$. Then the map $\phi := \mu(-, 1_R) : S \rightarrow R$ is a ring map by bilinearity of the multiplication map:

$$\phi(s + s') = \mu(s + s', 1_R) = \mu(s, 1_R) + \mu(s', 1_R) = \phi(s) + \phi(s')$$

and

$$\phi(1_S) = \mu(1_S, 1_R) = 1_R,$$

and

$$\phi(s)\phi(s') = m(s \cdot 1_R, s' \cdot 1_R) = ss' \cdot m(1_R, 1_R) = \phi(ss').$$

□

PROPOSITION 13.13. Fix S and two S -algebras R_1, R_2 . TFAE:

(1) A ring map $f : R_1 \rightarrow R_2$ such that the diagram

$$\begin{array}{ccc} & S & \\ \swarrow & & \searrow \\ R_1 & \xrightarrow{\quad} & R_2 \end{array}$$

commutes.

(2) A ring map $f : R_1 \rightarrow R_2$ which is a map of S -modules.

DEFINITION 13.14. A ring map $R_1 \rightarrow R_2$ satisfying either (hence both) of the above conditions is called a map of S -algebras.

PROOF. (1) \implies (2). The S -module structure is given by the composition

$$S \times R \rightarrow R \times R \rightarrow R$$

where the first map is the product $\phi \times \text{id}_R$. Behold the commutativity of the following diagram:

$$\begin{array}{ccc}
 S \times R_1 & \xrightarrow{\text{id}_S \times f} & S \times R_2 \\
 \phi_1 \times \text{id}_{R_1} \downarrow & & \downarrow \phi_2 \times \text{id}_{R_2} \\
 R_1 \times R_1 & \xrightarrow{f \times f} & R_2 \times R_2 \\
 \downarrow & & \downarrow \\
 R_1 & \xrightarrow{f} & R_2
 \end{array}$$

where the top rectangle commutes because the triangle in (1) does, and the bottom rectangle commutes by definition of ring map. This implies the big rectangle commutes, which is the definition of f being a module map.

(2) \implies (1). Consider the following commutative rectangles:

$$\begin{array}{ccc}
 S \times \{1_{R_1}\} & \xrightarrow{\text{id}_S} & S \times \{1_{R_2}\} \\
 \downarrow & & \downarrow \\
 S \times R_1 & \xrightarrow{\text{id}_S \times f} & S \times R_2 \\
 \mu \downarrow & & \downarrow \mu \\
 R_1 & \xrightarrow{f} & R_2
 \end{array}$$

The top rectangle commutes because ring maps send 1 to 1, and the bottom commutes by definition of module map. This means the outer rectangle commutes, but the vertical compositions are precisely the definition of the ring maps $\phi_i : S \rightarrow R_i$ in our first proposition. The commutativity of this outer rectangle is precisely the commutativity of the triangle in (1). \square

Using this language, we can rephrase the Theorem 13.6 as follows:

THEOREM 13.15. Let R_1, R_2 be S -algebras. Then there exists an S -algebra $R_1 \otimes_S R_2$, together with S -algebra maps

$$R_1 \rightarrow R_1 \otimes_S R_2, \quad R_2 \rightarrow R_1 \otimes_S R_2,$$

such that the following universal property is satisfied:

For any S -algebra W equipped with S -algebra maps $R_1 \rightarrow W, R_2 \rightarrow W$, there exists a unique S -algebra map $j : R_1 \otimes_S R_2 \rightarrow W$ making

the following diagram commute:

$$\begin{array}{ccc}
 & R_1 & \\
 & \downarrow & \searrow \\
 R_2 \longrightarrow & R_1 \otimes_S R_2 & \\
 & \searrow & \downarrow \\
 & & W
 \end{array}$$

REMARK 13.16. Note that the diagram above is the exact same shape as the universal property diagram for direct sum:

$$\begin{array}{ccc}
 & V_1 & \\
 & \downarrow & \searrow \\
 V_2 \longrightarrow & V_1 \oplus V_2 & \\
 & \searrow & \downarrow \\
 & & W
 \end{array}$$

where the V_i are vector spaces over some fixed field k and W is, too. This is to say: Direct sum is the coproduct in the category of k -vector spaces, while \otimes_S is the coproduct in the category of S -algebras.

PROOF OF THEOREM 13.6 AND THEOREM 13.15 (THEY'RE THE SAME THING). We know a unique S -module map exists by definition of the tensor product. We just need to show it is also a ring map.

What is the ring structure on $R_1 \otimes_S R_2$? We define it as follows:

$$\left(\sum_{i,j} r_i \otimes r_j \right) \left(\sum_{a,b} r_a \otimes r_b \right) := \sum_{i,j,a,b} r_i r_a \otimes r_j r_b.$$

To see this is well-defined, just use the universal property of tensor product \otimes_S to see that the function arises from a map linear in each of the variables $R_1 \times R_2 \times R_1 \times R_2$. It is easily checked, then, that given the ring maps $h_i : R_i \rightarrow W$, the assignment

$$r_1 \otimes r_2 \mapsto h_1(r_1)h_2(r_2)$$

is a ring map. □