Lecture 15: Fun example: Cobordisms.

Most of what we covered is in the TeXed notes for Lecture 14.

However, I did cover one new topic: Cobordism categories.

So far, all the categories we've studied look like "sets" underlie them. In particular, in $\mathcal{C}$ = groups, rings, vector spaces, sets, all the morphisms $f : X \to Y$ are always functions satisfying some property.

Well, here is an example where the category's morphisms are *not* of that flavor.

DEFINITION 15.1. Let Cob_1^{or} be the category of *oriented, 1-dimensional cobordisms.*

(1) An object X is given by a finite subset of $\mathbb{R}^\infty$, each element given a sign of plus or minus. One should think of this as a collection of points floating in space, each point with an orientation.

(2) A morphism from X_0 to X_1 is the data of an subset $\Gamma \subset \mathbb{R}^\infty \times [0,1]$, with the data of an orientation, satisfying the following conditions:
   (a) Γ is a disjoint union of smooth curves, possibly with boundary. We demand that the boundary of Γ is precisely the intersection of Γ with $\mathbb{R}^\infty \times \{0\} \cup \mathbb{R}^\infty \times \{1\}$.
   (b) We demand that the boundary of Γ at 0 is precisely X_0, and the boundary of Γ at 1 is precisely X_1. These must be compatible with the orientations.
   (c) We declare two Γ to be equal if one can be smoothly transformed (isotoped) into the other while respecting boundaries.

(3) Composition: If $\Gamma : X_0 \to X_1$ and $\Gamma' : X_1 \to X_2$, the composition is given by gluing Γ and Γ' along X_1. Note that this naturally lives over the interval $[0,2]$, but we can reparametrize this interval. This is compatible with the isotopy equivalence relation above.

Some examples of objects: The empty subset, a singleton subset with positive orientation, a singleton subset with negative orientation, and disjoint unions of these.

Some examples of morphisms: Γ could be

(1) A circle, which has no boundary; this is a morphism from $\emptyset$ to $\emptyset$.
(2) A horseshoe, with boundary only on $\mathbb{R}^\infty \times \{0\}$. Necessarily, the boundary will consist of a $+$ point and a $-$ point.
(3) A co-horseshoe, with boundary only on $\mathbb{R}^\infty \times \{1\}$. Necessarily, the boundary will consist of a $+$ point and a $-$ point.
(4) A single line interval with one boundary point on $\mathbb{R}^\infty \times \{0\}$ and the other on $\mathbb{R}^\infty \times \{1\}$. Necessarily, these two boundary points have the same orientation; this is the identity morphism from the boundary point to itself.

Theorem 15.2. Let $Z : \mathsf{Cob}_1^{or} \to \mathsf{Vect}_k$ be a functor taking $\coprod$ to $\otimes_k$. Then $Z(*_+) := V_+$ is finite dimensional, and one can identify $Z(*_-)$ with its dual.

We'll articulate this more accurately next lecture.