## Lecture 17: Limits, colimits, and some rings

## 17.1. Colimits.

DEFINITION 17.1. A *diagram* in  $\mathcal{C}$  is a functor  $F : \mathcal{D} \to \mathcal{C}$ . We say F is a diagram of shape  $\mathcal{D}$ .

EXAMPLE 17.2. Let  $\mathcal{D} = * \coprod *$  be the category with two objects and only identity morphisms. A diagram in the shape of  $\mathcal{D}$  picks out two objects X, X' of  $\mathcal{C}$ .

DEFINITION 17.3. Fix a category  $\mathcal{D}$ . The (left) *cone* category on  $\mathcal{D}$ , denoted

 $\mathcal{D}^{\triangleright}$ 

is the category where

(1) 
$$\operatorname{Ob} \mathcal{D}^{\triangleright} := \operatorname{Ob} \mathcal{D} \coprod \{*\}$$
  
(2)

$$\hom_{\mathcal{D}^{\triangleright}}(X,Y) := \begin{cases} \hom_{\mathcal{D}}(X,Y) & X, Y \in \operatorname{Ob} \mathcal{D} \\ pt & Y = * \\ \emptyset & \text{otherwise} \end{cases}$$

Note, for instance, that even if  $\mathcal{D}$  already has a terminal object,  $\mathcal{D}^{\triangleright}$  has a new terminal object, and it is not isomorphic to the original terminal object of  $\mathcal{D}$ .

Note also that composition is forced upon you, as the only new morphism spaces are empty or are singletons.

EXAMPLE 17.4. With  $\mathcal{D}$  as above,  $\mathcal{D}^{\triangleright}$  looks as follows:



DEFINITION 17.5. Fix a diagram  $F : \mathcal{D} \to \mathcal{C}$ . Then define

$$\mathsf{Fun}_{\mathcal{D}}(\mathcal{D}^{\triangleright},\mathcal{C})$$

to be the category where

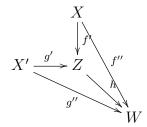
- (1) An object is a functor  $F' : \mathcal{D}^{\triangleright} \to \mathcal{C}$  such that  $F'|_{\mathcal{D}} = F$ ; i.e., the restriction to  $\mathcal{D}$  is the original diagram.
- (2) A morphism is a natural transformation  $\eta: F' \to F''$  such that  $\eta_{\mathcal{D}} = \mathrm{id}_F$ ; i.e., the restriction to  $\mathcal{D}$  is just the identity natural transformation.

REMARK 17.6. The notation does not indicate the dependence on F.

EXAMPLE 17.7. Continuing the previous example, an object of  $\operatorname{Fun}_{\mathcal{D}}(\mathcal{D}^{\triangleright}, \mathcal{C})$  picks out a diagram of the shape



and a morphism in this category picks out a commutative diagram as follows:



DEFINITION 17.8. Fix a category  $\mathcal{E}$ . An *initial object* in  $\mathcal{E}$  is an object X such that hom(X, Y) = pt for any  $Y \in \mathcal{E}$ .

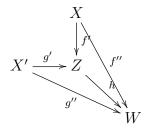
Note any two initial objects are isomorphic.

DEFINITION 17.9. Fix a diagram  $F : \mathcal{D} \to \mathcal{C}$ . A *colimit* for F is an initial object of the category  $\operatorname{Fun}_{\mathcal{D}}(\mathcal{D}^{\triangleright}, \mathcal{C})$ .

EXAMPLE 17.10. Continuing the previous example, an initial object of  $\operatorname{\mathsf{Fun}}_{\mathcal{D}}(\mathcal{D}^{\triangleright}, \mathcal{C})$  is some diagram



such that for any other diagram (below indicated using f'', g'', W) there is a *unique* morphism  $h: Z \to W$  making the following commute:



DEFINITION 17.11. A colimit in the shape of  $\mathcal{D} = * \coprod *$  is called a *coproduct* in  $\mathcal{C}$ .

17.2. Limits. One can likewise define limits as an initial object in  $\operatorname{Fun}_{\mathcal{D}^{\operatorname{op}}}((\mathcal{D}^{\operatorname{op}})^{\triangleright}, \mathcal{C}^{\operatorname{op}}).$ 

But this is opaque. Here are the dual constructions to define limits, spelled out:

DEFINITION 17.12. Fix a category  $\mathcal{D}$ . The (right) *cone* category on  $\mathcal{D}$ , denoted

 $\mathcal{D}^{\triangleleft}$ 

is the category where

(1) 
$$\operatorname{Ob} \mathcal{D}^{\triangleleft} := \{*\} \coprod \operatorname{Ob} \mathcal{D}$$
  
(2)  
 $\operatorname{hom}_{\mathcal{D}^{\triangleleft}}(X, Y) := \begin{cases} \operatorname{hom}_{\mathcal{D}}(X, Y) & X, Y \in \operatorname{Ob} \mathcal{D} \\ pt & X = * \\ \emptyset & \text{otherwise} \end{cases}$ 

EXAMPLE 17.13. With  $\mathcal{D}$  as above,  $\mathcal{D}^{\triangleleft}$  looks as follows:



DEFINITION 17.14. Fix a diagram  $F : \mathcal{D} \to \mathcal{C}$ . Then define

 $\operatorname{Fun}_{\mathcal{D}}(\mathcal{D}^{\triangleleft}, \mathcal{C})$ 

to be the category where

- (1) An object is a functor  $F' : \mathcal{D}^{\triangleleft} \to \mathcal{C}$  such that  $F'|_{\mathcal{D}} = F$ ; i.e., the restriction to  $\mathcal{D}$  is the original diagram.
- (2) A morphism is a natural transformation  $\eta: F' \to F''$  such that  $\eta_{\mathcal{D}} = \mathrm{id}_F$ ; i.e., the restriction to  $\mathcal{D}$  is just the identity natural transformation.

DEFINITION 17.15. Fix a category  $\mathcal{E}$ . A terminal object in  $\mathcal{E}$  is an object Y such that hom(X, Y) = pt for any  $X \in \mathcal{E}$ .

DEFINITION 17.16. Fix a diagram  $F : \mathcal{D} \to \mathcal{C}$ . A *limit* for F is a terminal object of the category  $\operatorname{Fun}_{\mathcal{D}}(\mathcal{D}^{\triangleleft}, \mathcal{C})$ .

TABLE 3. Some notation. Note that the word "limit" can be used to describe both limits and colimits in the literature; the words "inverse" or "directed" give indication of whether one's talking about limits or colimits.

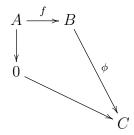
Colimits	Limits
$\operatorname{colim}(F:\mathcal{D}\to\mathcal{D})$	$\lim(F:\mathcal{D}\to\mathcal{C})$
$\operatorname{colim}_{\mathcal{D}} F$	$\lim_{\mathcal{D}} F$
$\operatorname{colim} F$	$\lim F$
$\lim\nolimits_{\to} \mathcal{D}$	$\lim_{\leftarrow} \mathcal{D}$
"directed limit"	"inverse limit"

## 17.3. Exercises.

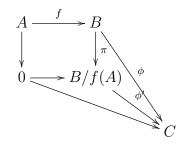
EXERCISE 17.17. Articulate the universal property of quotients of R-modules using colimits.

EXERCISE 17.18. Articulate the *p*-adic integers as a limit in rings.

Solutions: Fix  $f : A \to B$  a map of *R*-modules and  $\pi : B \to B/f(A)$ the quotient map. The universal property of quotients B/f(A) says that for any map  $\phi : B \to C$  of *R*-modules for which ker  $\phi \supset f(A)$ , there is a unique morphism  $\phi' : B/f(A) \to C$  such that  $\phi' \circ \pi = \phi$ . But the requirement ker  $\phi \supset f(A)$  is the expressing the commutativity of the following diagram:



And the universal property is expressing the uniqueness of  $\phi'$  in the commutative diagram below:



So  $\mathcal{D} = * \leftarrow * \to *$  is the shape, and any functor  $\mathcal{D} \to RMod$  looking like



has a colimit given by the quotient B/f(A) (equipped with the quotient map  $B \to B/f(A)$ ).

As for the next exercise, the *p*-adics can be written as a *limit* 

$$\ldots \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$$

of a functor from  $\mathcal{D} = (\mathbb{Z}_{\leq 0}, \leq)$  to Rings.

Analogously, we have the sequence of rings

$$\ldots \to \mathbb{C}[x]/x^3 \to \mathbb{C}[x]/x^2 \to \mathbb{C}[x]/x \cong \mathbb{C}$$

whose limit is  $\mathbb{C}[[x]]$ , the ring of power series. This sequence has a geometric interpretation:  $\mathbb{C}[x]/x \cong \mathbb{C}$  is the functions on the origin in  $\mathbb{A}^1$  (the complex line), and  $\mathbb{C}[x]/x^n$  is the (n-1)st order neighborhood