Lecture 17: Limits, colimits, and some rings

17.1. Colimits.

DEFINITION 17.1. A *diagram* in $\mathcal{C}$ is a functor $F : \mathcal{D} \to \mathcal{C}$. We say F is a diagram of shape $\mathcal{D}$.

EXAMPLE 17.2. Let $\mathcal{D} = * \coprod *$ be the category with two objects and only identity morphisms. A diagram in the shape of $\mathcal{D}$ picks out two objects X, X' of $\mathcal{C}$.

DEFINITION 17.3. Fix a category $\mathcal{D}$. The (left) *cone* category on $\mathcal{D}$, denoted

$$\mathcal{D}^{\triangleright}$$

is the category where

(1) $\mathrm{Ob}\,\mathcal{D}^{\triangleright} := \mathrm{Ob}\,\mathcal{D} \coprod \{*\}$
(2)

$$\hom_{\mathcal{D}^{\triangleright}}(X,Y) := \begin{cases} \hom_{\mathcal{D}}(X,Y) & X, Y \in \mathrm{Ob}\,\mathcal{D} \\ pt & Y = * \\ \emptyset & \text{otherwise} \end{cases}$$

Note, for instance, that even if $\mathcal{D}$ already has a terminal object, $\mathcal{D}^{\triangleright}$ has a new terminal object, and it is not isomorphic to the original terminal object of $\mathcal{D}$.

Note also that composition is forced upon you, as the only new morphism spaces are empty or are singletons.

EXAMPLE 17.4. With $\mathcal{D}$ as above, $\mathcal{D}^{\triangleright}$ looks as follows:

$$\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & *. \end{tikzcd}$$

Definition 17.5. Fix a diagram $F : \mathcal{D} \to \mathcal{C}$. Then define

$$\mathsf{Fun}_{\mathcal{D}}(\mathcal{D}^{\triangleright}, \mathcal{C})$$

to be the category where

(1) An object is a functor $F' : \mathcal{D}^{\triangleright} \to \mathcal{C}$ such that $F'|_{\mathcal{D}} = F$; i.e., the restriction to $\mathcal{D}$ is the original diagram.
(2) A morphism is a natural transformation $\eta : F' \to F''$ such that $\eta_{\mathcal{D}} = \mathrm{id}_F$; i.e., the restriction to $\mathcal{D}$ is just the identity natural transformation.

Remark 17.6. The notation does not indicate the dependence on F.

Example 17.7. Continuing the previous example, an object of $\mathsf{Fun}_{\mathcal{D}}(\mathcal{D}^{\triangleright}, \mathcal{C})$ picks out a diagram of the shape

$$\begin{tikzcd} & X \arrow[d, "f'"] \\ X' \arrow[r, "g'"] & Z \end{tikzcd}$$

and a morphism in this category picks out a commutative diagram as follows:

$$\begin{tikzcd} & X \arrow[d, "f'"] \arrow[ddr, "f''"] & \\ X' \arrow[r, "g'"] \arrow[drr, "g''"'] & Z \arrow[dr, "h"] & \\ & & W \end{tikzcd}$$

Definition 17.8. Fix a category $\mathcal{E}$. An *initial object* in $\mathcal{E}$ is an object X such that $\hom(X, Y) = pt$ for any $Y \in \mathcal{E}$.

Note any two initial objects are isomorphic.

Definition 17.9. Fix a diagram $F : \mathcal{D} \to \mathcal{C}$. A *colimit* for F is an initial object of the category $\mathsf{Fun}_{\mathcal{D}}(\mathcal{D}^{\triangleright}, \mathcal{C})$.

EXAMPLE 17.10. Continuing the previous example, an initial object of $\mathsf{Fun}_{\mathcal{D}}(\mathcal{D}^{\triangleright}, \mathcal{C})$ is some diagram

$$\begin{tikzcd} & X \arrow[d, "f'"] \\ X' \arrow[r, "g'"] & Z \end{tikzcd}$$

such that for any other diagram (below indicated using f'', g'', W) there is a *unique* morphism $h : Z \to W$ making the following commute:

$$\begin{tikzcd} & X \arrow[d, "f'"] \arrow[ddr, "f''"] & \\ X' \arrow[r, "g'"] \arrow[drr, "g''"'] & Z \arrow[dr, "h"] & \\ & & W \end{tikzcd}$$

DEFINITION 17.11. A colimit in the shape of $\mathcal{D} = * \coprod *$ is called a *coproduct* in $\mathcal{C}$.

17.2. Limits. One can likewise define limits as an initial object in

$$\mathsf{Fun}_{\mathcal{D}^{\mathrm{op}}}((\mathcal{D}^{\mathrm{op}})^{\triangleright}, \mathcal{C}^{\mathrm{op}}).$$

But this is opaque. Here are the dual constructions to define limits, spelled out:

DEFINITION 17.12. Fix a category $\mathcal{D}$. The (right) *cone* category on $\mathcal{D}$, denoted

$$\mathcal{D}^{\triangleleft}$$

is the category where

(1) $\operatorname{Ob} \mathcal{D}^{\triangleleft} := \{*\} \coprod \operatorname{Ob} \mathcal{D}$
(2)

$$\hom_{\mathcal{D}^{\triangleleft}}(X, Y) := \begin{cases} \hom_{\mathcal{D}}(X, Y) & X, Y \in \operatorname{Ob} \mathcal{D} \\ pt & X = * \\ \emptyset & \text{otherwise} \end{cases}$$

EXAMPLE 17.13. With $\mathcal{D}$ as above, $\mathcal{D}^{\triangleleft}$ looks as follows:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & * \\ *. & \end{tikzcd}$$

DEFINITION 17.14. Fix a diagram $F : \mathcal{D} \to \mathcal{C}$. Then define

$$\mathsf{Fun}_{\mathcal{D}}(\mathcal{D}^{\triangleleft}, \mathcal{C})$$

to be the category where

(1) An object is a functor $F' : \mathcal{D}^{\triangleleft} \to \mathcal{C}$ such that $F'|_{\mathcal{D}} = F$; i.e., the restriction to $\mathcal{D}$ is the original diagram.
(2) A morphism is a natural transformation $\eta : F' \to F''$ such that $\eta_{\mathcal{D}} = \mathrm{id}_F$; i.e., the restriction to $\mathcal{D}$ is just the identity natural transformation.

DEFINITION 17.15. Fix a category $\mathcal{E}$. A *terminal object* in $\mathcal{E}$ is an object Y such that $\hom(X, Y) = pt$ for any $X \in \mathcal{E}$.

DEFINITION 17.16. Fix a diagram $F : \mathcal{D} \to \mathcal{C}$. A *limit* for F is a terminal object of the category $\mathsf{Fun}_{\mathcal{D}}(\mathcal{D}^{\triangleleft}, \mathcal{C})$.

TABLE 3. Some notation. Note that the word "limit" can be used to describe both limits and colimits in the literature; the words "inverse" or "directed" give indication of whether one's talking about limits or colimits.

Colimits	Limits
$\mathrm{colim}(F : \mathcal{D} \to \mathcal{D})$	$\lim(F : \mathcal{D} \to \mathcal{C})$
$\mathrm{colim}_{\mathcal{D}} F$	$\lim_{\mathcal{D}} F$
$\mathrm{colim}\, F$	$\lim F$
$\lim_{\to} \mathcal{D}$	$\lim_{\leftarrow} \mathcal{D}$
"directed limit"	"inverse limit"

17.3. Exercises.

EXERCISE 17.17. Articulate the universal property of quotients of R-modules using colimits.

EXERCISE 17.18. Articulate the p-adic integers as a limit in rings.

Solutions: Fix $f : A \to B$ a map of R-modules and $\pi : B \to B/f(A)$ the quotient map. The universal property of quotients $B/f(A)$ says that for any map $\phi : B \to C$ of R-modules for which $\ker \phi \supset f(A)$, there is a unique morphism $\phi' : B/f(A) \to C$ such that $\phi' \circ \pi = \phi$.

But the requirement $\ker \phi \supset f(A)$ is the expressing the commutativity of the following diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[ddr, "\phi"] & \\
0 \arrow[drr] & & \\
& & C
\end{tikzcd}$$

And the universal property is expressing the uniqueness of ϕ' in the commutative diagram below:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d, "\pi"] \arrow[dr, "\phi"] & \\
0 \arrow[r] \arrow[rr, bend right] & B/f(A) \arrow[r, "\phi'"] & C
\end{tikzcd}$$

So $\mathcal{D} = * \leftarrow * \rightarrow *$ is the shape, and any functor $\mathcal{D} \to RMod$ looking like

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \\
0 &
\end{tikzcd}$$

has a colimit given by the quotient $B/f(A)$ (equipped with the quotient map $B \to B/f(A)$).

As for the next exercise, the p-adics can be written as a *limit*

$$\ldots \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$$

of a functor from $\mathcal{D} = (\mathbb{Z}_{\leq 0}, \leq)$ to Rings.

Analogously, we have the sequence of rings

$$\ldots \to \mathbb{C}[x]/x^3 \to \mathbb{C}[x]/x^2 \to \mathbb{C}[x]/x \cong \mathbb{C}$$

whose limit is $\mathbb{C}[[x]]$, the ring of power series. This sequence has a geometric interpretation: $\mathbb{C}[x]/x \cong \mathbb{C}$ is the functions on the origin in $\mathbb{A}^1$ (the complex line), and $\mathbb{C}[x]/x^n$ is the $(n-1)$st order neighborhood