Some Recollections:

- Let R be a $\mathbb{k}$-algebra. Then $\forall$ $a \in R$, $\exists!$ $\mathbb{k}$-alg. map

$$\begin{aligned} \mathbb{k}[x] &\longrightarrow R \\ x &\longmapsto a. \end{aligned}$$

(Univ. property of $\mathbb{k}[x]$).

<u>Defn</u> a is called <u>algebraic</u> (over $\mathbb{k}$) if the above map has non-trivial kernel.

(ie, $\exists$ polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ st $f(a) = 0$.)

<u>Propn</u> If R is also an integral domain, then any algebraic element of R has a mult. inverse.

<u>Pf</u> Let I be the kernel of $\mathbb{k}[x] \longrightarrow R$, $x \longmapsto a$, with a algebraic. Since R has no zero divisors, $\mathbb{k}[x]/I$ doesn't either — hence I is prime. But $I \neq (0)$ by assumption, so I is maximal. //

[JUST IN CASE:] <u>Propn</u> R PID $\Rightarrow$ any non-zero prime ideal is maximal.

<u>Pf</u>: Let (x) be prime, $(x) \subset (y)$. Then $x = ay$; but (x) is prime, so $ay \in (x) \Rightarrow a \in (x)$ or $y \in (x)$. Note $a \in (x) \Rightarrow a = a'x$

$\Rightarrow (1 - a'y)x = 0$

Goals today:

Thm ① Let $\mathbb{k}$ be alg. closed.
Then

$$\begin{aligned} \operatorname{MaxSpec}(\mathbb{k}[x_1, \dots, x_n]) &\longrightarrow \mathbb{k}^n \\ M &\longmapsto (ev(x_1), \dots, ev(x_n)) \end{aligned}$$

where

$$\begin{aligned} ev: \mathbb{k}[x_1, \dots, x_n] &\xrightarrow{\pi} \mathbb{k}[x_1, \dots, x_n]/M \cong \mathbb{k} \\ f &\longmapsto \pi(f) \end{aligned}$$

is "evaluation of f @ M," is a bijection.
The inverse map is $(a_1, \dots, a_n) \longmapsto (x_1 - a_1, \dots, x_n - a_n) \subset \mathbb{k}[x_1, \dots, x_n]$.

Thm ② Let $\mathbb{k}$ be alg. closed, and $S \subset \mathbb{k}[x_1, \dots, x_n]$ a subset. Let

$$V(S) := \{ (a_1, \dots, a_n) \in \mathbb{k}^n \mid f(a_1, \dots, a_n) = 0 \ \forall f \in S \}.$$

Then

$$\operatorname{MaxSpec}\left(\mathbb{k}[x_1, \dots, x_n]/(S) \right) \xrightarrow{ev} V(S)$$

is a bijection.

Pf of Thm ①

Assume $m \subset \mathbb{k}[x_1, \dots, x_n]$ maximal. Then

$$R/m \xrightarrow{\cong} F \quad \leftarrow \text{some field}$$

and the composite

$$\mathbb{k} \hookrightarrow R \longrightarrow R/m \xrightarrow{\cong} F$$

is a field extension. (ie, it exhibits $\mathbb{k}$ as a subfield of F).

$\boxed{\mathrm{I}}$ By construction, any element of F is algebraic over $\mathbb{k}$. (i.e., if $\alpha \in F$, the induced map $\mathbb{k}[t] \longrightarrow F$, $t \longmapsto \alpha$, has non-trivial kernel.), so because $\mathbb{k}$ is algebraically closed, the map $\mathbb{k} \longrightarrow F$ is an isomorphism.

Take-away: $\mathbb{k}$ alg closed $\Rightarrow$ $R/m \cong \mathbb{k}$, $\forall$ m maximal, $R = \mathbb{k}[x_1, \dots, x_n]$.

Now, if m maximal, consider $\phi_i : \mathbb{k}[x_i] \hookrightarrow \mathbb{k}[x_1, \dots, x_n]$.

$\boxed{\mathrm{II}}$ We claim $\phi_i^{-1}(m) \subset \mathbb{k}[x_i]$ is also maximal, hence

$$\mathbb{k}[x_i] / \phi_i^{-1}(m) \cong \mathbb{k}, \quad \text{and} \quad \phi_i^{-1}(m) = (x_i - a_i)$$

(b/c $\mathbb{k}$ is alg closed, any max. ideal of $\mathbb{k}[t]$ is gen. by a linear polynomial.)

Thus, we conclude

$$(x_1-a_1, \dots, x_n-a_n) \subset m.$$

$\boxed{\text{III}}$ Observing that $(x_1-a_1, \dots, x_n-a_n)$ is maximal for any choice of $(a_1, \dots, a_n)$, we're finished.

Pf of Thm ② Recall the correspondence

$$\{\text{ideals } I \supset S\} \simeq \{\text{ideals of } R/(S)\}.$$

$$I \longmapsto \pi(I)$$
$$\pi^{-1}(J) \longleftarrow J$$
$$\pi : R \to R/(S).$$

This obviously respects $\subseteq$, hence maximality. Hence we just need to show

$$(a_1, \dots, a_n) \in V(S) \iff (x_1-a_1, \dots, x_n-a_n) \supset S.$$

This is obvious since both conditions are equivalent to S being in the kernel of $ev_{a_1, \dots, a_n}$. //

So we have to prove three lemmas.

Lemma $\boxed{I}$ Suppose R a $\mathbb{k}$-alg and a field.
Suppose $\exists$ S, finitely-gen. $\mathbb{k}$-alg, int. domain, s.t. $R \subseteq S$.
Then any elmt of R is algebraic over $\mathbb{k}$; in particular,
$R/\mathbb{k}$ is a finite field extension.
(And if $\mathbb{k}$ is alg closed, $R \cong \mathbb{k}$.)

This implies claim $\boxed{I}$.

Pf of Lemma $\boxed{I}$: By contradiction.

Suppose $\exists\ \alpha_1 \in R$ NOT alg over $\mathbb{k}$.
Choose generators $\alpha_2, \cdots, \alpha_n$ of S as $\mathbb{k}$-algebra.
Then we have a map

$$\mathbb{k}[x_1, \cdots, x_n] \longrightarrow S, \qquad x_i \longmapsto \alpha_i$$

which is a surjection. Note $\mathbb{k}[x_1] \longrightarrow S$ is an injection since α_1 is assumed non-algebraic.

Rearranging if necessary, assume $\{\alpha_i\}_{i=1,\cdots,r}$, $r \le n$, is the maximal cllxn of α_i s.t.

$$\mathbb{k}[x_1, \cdots, x_r] \longrightarrow S$$

is an injection. (ie, $\alpha_1, \cdots, \alpha_r$ are algebraically independent).
Call the image S', and consider the induced map

$$\mathrm{Quot}(S') \hookrightarrow \mathrm{Quot}(S)$$

which is a finite extension of fields! Rename fields $K \hookrightarrow L$.

Then any $x \in S$ acts on $\mathrm{Quot}(S)$ as a K-linear endomorphism, so get

$$S \xrightarrow[\rho]{} M_{\ell \times \ell}(K)$$

by choosing some K-basis for L. (This is a ring map).

Each $\rho(\alpha_i)$ has matrix entries in $K = Quot(S')$ of the form $\frac{s_1}{s_2}$, $s_i \in S'$.
Let g be the least common multiple of the denominators s_2.

Finally, consider the element $\alpha_1^{-1} = \frac{1}{\alpha_1} \in Quot(S')$. (Remember, α_1 is the non-algebraic elmnt.) Since g is a multiple of any denominator, $\exists$ $f \in S'$ s.t.

$$gf = \alpha_1 .$$

Recall $S' \cong \Bbbk[x_1, \ldots, x_l]$ is a UFD; $gf = \alpha_1$ means the factorization of g into irreducibles contains factors in $\Bbbk[\alpha_1] \subset S'$.
Let $p_1, \ldots, p_r$ be all the irreducibles of $\Bbbk[\alpha_1]$ appearing in the factorization of g. We arrive at a contradiction as follows:

Any non-unit element x of $\Bbbk[\alpha_1]$ has $\rho(\frac{1}{x}) = \mathrm{diag}(\frac{1}{x}, \ldots, \frac{1}{x})$. Then the factors of g must divide x (as in the argument for α_1), but if we take

$$x = \left(\prod_{i=1}^{r} p_i\right) + 1$$

we have our contradiction.

Summary: The existence of non-alg α implied the existence of a finite collxn of irreducibles in a polynomial ring that factors any non-unit. This is impossible (for the same reason $\mathbb{Z}$ has infinitely many primes).

<u>Lemma</u> $\boxed{\text{II}}$: Let B be a finitely generated $\mathbb{k}$-algebra. Then for any $\mathbb{k}$-algebra map $\phi: A \to B$, ϕ^{-1} takes maximal ideals to maximal ideals.

<u>Pf</u>: We have the injections

$$A/\phi^{-1}(m) \hookrightarrow B/m$$

whenever m is an ideal. If m is maximal, this is an inclusion of an integral domain into a field. By Lemma $\boxed{\text{I}}$, B/m is algebraic over $\mathbb{k}$; so given any $0 \neq \alpha \in A/\phi^{-1}(m)$, we know that

$$\begin{array}{ccccc} \mathbb{k}[x] & \xrightarrow{\pi_\alpha} & A/\phi^{-1}(m) & \hookrightarrow & B/m \\ x & \longmapsto & \alpha & & \end{array}$$

has $\underbrace{\text{kernel}}_{\text{(non-trivial)}}$; since $\mathbb{k}[x]$ is a PID, image(π_α) being an integral domain implies image(π_α) is a field — ie, α has a mult. inverse. This shows $A/\phi^{-1}(m)$ is a field, hence $\phi^{-1}(m)$ is maximal. //

<u>Rmk</u> More generally, we just showed that if R is an integral domain, any algebraic element of R has a mult. inverse in R.

Lemma $\boxed{\text{III}}$ Let $\Bbbk$ be any field. Then the ideal $(x_1-a_1, \dots, x_n-a_n)$ is maximal in $\Bbbk[x_1, \dots, x_n]$, for any choice $(a_1, \dots, a_n) \in \Bbbk$.

Pf: Call the ideal I. Then

$$ev_{a_1,\dots,a_n} : f \longmapsto f(a_1,\dots,a_n)$$

shows that any $f \in \Bbbk[x_1,\dots,x_n]$ is congruent to an element in $\Bbbk$ (mod I). Thus we have

$$\begin{tikzcd} \Bbbk \arrow[r, hook, "\text{constants}"] \arrow[dr, "\phi"'] & \Bbbk[x_1,\dots,x_n] \arrow[r, "ev_{a_1,\dots,a_n}"] \arrow[d, "\pi"] & \Bbbk \\ & \Bbbk[x_1,\dots,x_n]/I \arrow[ur, dashed, "\psi"', "\exists"] & \end{tikzcd}.$$

$\psi \circ \phi = id_{\Bbbk}$ shows ϕ is an injection, while we know ϕ is a surjection by the second sentence of our proof. //