Today: Two Nullstellensatz (again)

So far, we've talked about a few theorems w/ geometric content.

Throughout, assume $\Bbbk$ alg. closed.

Thm① $\exists$ bijection

$$\mathrm{MaxSpec}(\Bbbk[x_1, \dots, x_n]) \cong \Bbbk^n.$$

This justifies (a) maximal ideals are "points,"
(b) $\Bbbk[x_1, \dots, x_n]$ corresponds to affine n-dim. space in the rings $\longleftrightarrow$ spaces dictionary.

Cor/Thm② Let $I \subset \Bbbk[x_1, \dots, x_n]$ and $V(I) = \{\vec{a} \in \Bbbk^n \text{ st } f(\vec{a}) = 0 \ \forall f \in I\}$ the vanishing locus of I. The above induces a bijection

$$\mathrm{MaxSpec}(\Bbbk[x_1, \dots, x_n]/I) \cong V(I).$$

This continues the justification of the ideas above, though (as we'll see) justification (b) is a bit feeble here.

Thm (Weak Nullstellensatz) $I \neq \Bbbk[x_1, \dots, x_n]$ proper ideal $\Rightarrow V(I) \neq \emptyset$.

This followed easily (as we saw last time). By Zorn's Lemma, I proper implies $R/I \neq 0$ has a maximal ideal. Hence by Thm②, $V(I) \neq \emptyset$.

However, we've talked before about the ambiguity in the correspondence

$$\text{ideals} \longleftrightarrow \text{subspaces}.$$

For example, the ideals $I=(x)$ and $J=(x^2)$ both have vanishing locus $V(I)=V(J)=\{0\}\subset \mathbb{k}$ given by the origin. (Here, the ring is $R=\mathbb{k}[x]$.)

Two ways to resolve this: (i) expand notion of "space" to include "schemes," where $J=(x^2)$ describes a space where maps from the space represent not just points in a target, but points w/ tangent vectors.

(ii) Understand when two ideals give rise to the same subspace.

We won't talk about schemes, so we pursue (ii). First, let's understand the "multiplicity" of (x^2) in (x).

Defn Fix a ring R and an ideal I.
The radical of I is the set

$$\sqrt{I} := \{f\in R \text{ s.t. } f^k\in I \text{ for some } k\geq 1\}.$$

Exer $\sqrt{I}$ is an ideal.

Pf. $f^k \in I \Rightarrow \forall x \in R$, $(xf)^k = x^k \cdot f^k \in I$ since I an ideal.
If $f^k, g^l \in I$, assume WLOG that $l \geq k$. Then

$$(f+g)^{2l} = \sum_{a+b=2l} \binom{2l}{a} f^a g^b$$

where $a \geq l \geq k$ or $b \geq l$. Hence (since I is an ideal) each summand is in I, so $(f+g)^{2l} \in I$. $\Rightarrow f+g \in \sqrt{I}$. //

Example $\sqrt{(x^2)} = (x)$.

The following says that this "multiplicity" is the only ambiguity in the dictionary:

Thm (Nullstellensatz)
Let $\Bbbk$ be algebraically closed, and $I \subset \Bbbk[x_1, \ldots, x_n]$ be an ideal.
Then

$$\mathcal{I}(V(I)) = \sqrt{I}.$$

Rmk Given $X \subset \Bbbk^n$, $\mathcal{I}(X) := \{ f \in \Bbbk[x_1, \ldots, x_n] \text{ s.t. } f|_X \equiv 0 \}$. This is, as usual, the ideal associated to a subspace.

Rmk This implies the weak Nullstellensatz: $V(I) = \phi \overset{\text{DEF}}{\Longrightarrow} \mathcal{I}(V(I)) = R$
$\overset{\text{Thm}}{\Longrightarrow} \sqrt{I} = R$
$\overset{\text{DEF}}{\Longrightarrow} 1^k \in I$ for some $k \geq 1$
$\Rightarrow 1 \in I$. //

(Hence the name "weak" Nullstellensatz.)

The proof we give of the Nullstellensatz depends on two lemmas:

<u>Lemma ①</u> For any ideal $I \subset R$,

$$\sqrt{I} = \bigcap_{\substack{I \subset p \\ p \text{ prime}}} p .$$

<u>Lemma ②</u> If $R = \mathbb{k}[x_1, \dots, x_n]$ where $\mathbb{k}$ is any field, then any prime ideal p is the intersection of the maximal ideals containing p:

$$p = \bigcap_{\substack{I \subset m \\ m \text{ maximal}}} m .$$

<u>Exercise</u> Using Lemmas and Thm ①, prove Nullstellensatz.

Soln

$$\sqrt{I} = \bigcap_{\substack{I \subset p \\ p \text{ prime}}} p \qquad \text{Lemma 1}$$

$$= \bigcap_{\substack{I \subset p \\ p \text{ prime}}} \bigcap_{\substack{p \subset m \\ m \text{ maximal}}} m \qquad \text{Lemma 2}$$

$$= \bigcap_{\substack{I \subset m \\ m \text{ maximal}}} m.$$

So it suffices to show that $\mathcal{I}(V(I)) = \bigcap_{\substack{I \subset m \\ m \text{ maximal}}} m.$

Observe: $f \in \mathcal{I}(V(I)) \iff f|_{V(I)} = 0$

$\iff \forall\, \vec{a} \in V(I),\ f \in \operatorname{Ker}(ev_{\vec{a}} : \mathbb{k}[x_1, \dots, x_n] \to \mathbb{k})$

$\iff \forall\, \vec{a} = (a_1, \dots, a_n) \in V(I),\ f \in (x_1 - a_1, \dots, x_n - a_n)$

moreover, $\vec{a} \in V(I) \iff g(\vec{a}) = 0\ \forall\, g \in I$

$\iff g \in \operatorname{Ker}(ev_{\vec{a}})\ \forall\, g \in I$

$\iff g \in m_{\vec{a}} = (x_1 - a_1, \dots, x_n - a_n)\ \forall\, g \in I$

$\iff I \subset m_{\vec{a}}.$

Combining we see $f \in \mathcal{I}(V(I)) \iff \forall\, m_{\vec{a}}$ st $I \subset m_{\vec{a}}$, $f \in m_{\vec{a}}$

$\iff \forall\, m$ st $I \subset m$, $f \in m$ Thm 1

ie, $\iff f \in \bigcap_{I \subset m} m.$ //

As usual, let's put some geometric context / intuition before proving these lemmas.

Exer Let R be a ring. Given $I \subset R$ an ideal, let

$$V(I) = \{ p \mid I \subset p \} \subset \mathrm{Spec}(R)$$

as usual. Show

$$V(I \cap J) = V(I) \cup V(J).$$

In words: intersection of ideals is union of subspaces
Intuitively: f vanishes along X and along $Y \iff f$ vanishes along $X \cup Y$.

Pf: $\subset$: If $p \in V(I \cap J)$, then $I \cap J \subset p$. Need to show $I \subset p$ or $J \subset p$.
If I is not contained in p, choose $f \in I \setminus (I \cap p)$. Then $\forall\, g \in J$,

$$fg \in I \cap J$$

by I, J being ideals. Since p is prime and $f \notin p$, $g \in p$.

$\supset$: If p contains I or J, it contains $I \cap J$. //

Via this exercise, lemma ② has the following interpretation: Any "subspace" is a union of its points, if that subspace corresponds to a prime ideal. And what kinds of subspaces do prime ideals correspond to?

(1) Subspaces without "infinitesimal" behavior (e.g., $f^k = 0 \Rightarrow f = 0$)
(2) "Irreducible" subspaces (see homework).

Such geometric objects are at least worth a name:

<u>Defn</u> A ring R is called a <u>Jacobson ring</u> if any prime ideal $p \subset R$ is the intersection of the maximal ideals containing p:

$$p = \bigcap_{\substack{p \subset m \\ m \text{ maximal}}} m.$$

You'll see a non-Jacobson ring in HW. Next time, we'll prove Lemma ② from a more general statement:

<u>Lemma</u> ②' If R is Jacobson, so is any finitely generated R-algebra.

Lemma ② follows because $\mathbb{k}$ is Jacobson if $\mathbb{k}$ is a field.

How does one interpret Lemma ①? Well, consider the quotient

$$R/I \longrightarrow R/\sqrt{I}.$$

A priori, R/I may have nilpotent elements — f s.t. $f^k = 0$.
Such subspaces can have the "infinitesimal" behavior of things like $k[x]/x^2$.
On the other hand, $R/\sqrt{I}$ is a ring we obtain precisely by eliminating such behavior.

So the passage from I to $\sqrt{I}$ is the passage from a space with possibly "fuzzy"/"infinitesimal" geometry to an "underlying" or "universal" space without that fuzziness.

Then, $\sqrt{I} = \bigcap_{I \subseteq p} p$ speaks to the following intuition: this unfuzzy version is the union of all the irreducible, unfuzzy subspaces contained in $\mathrm{Spec}(R/I)$.

Let's start by proving Lemma ①:

<u>Prop'n</u> Fix $U \subset R$ mult. closed.
If $I \subset R$ is maximal among
ideals $\{I' \text{ st } I' \cap U = \emptyset\}$
then I is prime.

<u>Rmk</u> Often, "if I is maximal among ..." produces a prime ideal. Geometrically, if I is maximal along "some algebraic condition", $\operatorname{Spec}(R/I)$ is minimal among "some geometric condition" so we tend to get "irreducible" subspaces (hence prime ideals — see HW).

<u>Rmk</u> In above prop'n, $I' \cap U = \emptyset$ means the fns $f \in I'$ remain uninvertible when we localize — ie, f vanishes somewhere on the complement. So the maximal I picks out some "minimal" subspace of the complement.

<u>Proof</u> Fix I maximal among I' st $I' \cap U = \emptyset$. Suppose $fg \in I$, and assume $f, g \notin I$. Then (by maximality) $(I+f)$, $(I+g)$ intersect U. So $\exists$
$$x + af,\ y + bg \in U$$
where $x, y \in I$, $f, g \in R$. But then
$$(x+af)(y+bg) = \underbrace{xy + afy + xbg}_{\text{in } I} + \underset{\text{in } I}{abfg} \in U$$
contradicting $I \cap U = \emptyset$. //

<u>Rmk</u> The proposition says nothing when $\{I'\} = \emptyset$ (eg. when $U \ni 0$).

Pf of Lemma ①

- $\sqrt{I} \subset \bigcap_{p \supset I} p$: $f \in \sqrt{I} \Rightarrow f^k \in I$ for some $k \geq 1$.
  If $I \subset p$ w/ p prime, $f^k \in p \Rightarrow f \in p$ (induct on k)
  Hence $f \in \bigcap_p p$.

- $\bigcap_{p \supset I} p \subset \sqrt{I}$: Suppose $f \notin \sqrt{I}$. Consider

$$\{ J' \subset R/I \mid J' \cap \{f^k\} = \phi \}.$$

(equivalently, consider those $J' \subset R$ ideals st $I \subset J'$ and $J' \cap \{f^k\} = \phi$.)
Since $\{f, f^2, \ldots\} = \{f^k\}$ is mult. closed, the maximal J by prop'n is prime, while maximal J exists by Zorn's lemma.
This exhibits a prime ideal NOT containing f, while containing I.
So $f \notin \sqrt{I} \Rightarrow f \notin \bigcap_{I \supset p} p$.