Review of last time:

Want to prove

Thm (Nullstellensatz) $\mathcal{I}(V(I)) = \sqrt{I}$
if $\mathbb{k}$ alg closed, $I \subset \mathbb{k}[x_1, \dots, x_n]$ ideal.

Proof relied on

Lemma ① $\sqrt{I} = \bigcap_{\substack{p \supset I \\ \text{prime}}} p$

Lemma ② $p \subset \mathbb{k}[x_1, \dots, x_n]$ (p prime, $\mathbb{k}$ any field)

$\Rightarrow p = \bigcap_{\substack{p \subset m \\ \text{maximal}}} m$.

Proof of Thm: $\sqrt{I} \overset{\text{Lemma ①}}{=} \bigcap_{p \supset I} p \overset{\text{Lemma ②}}{=} \bigcap_{p \supset I} \bigcap_{m \supset p} m = \bigcap_{I \supset m} m \overset{\text{Thm ①}}{=} \mathcal{I}(V(I))$. //

See notes from last time for proof of Lemma ①. Recall also:

Defn R is Jacobson if $\forall$ p prime, $p = \bigcap_{\substack{p \subset m \\ \text{maximal}}} m$.

Because any field is Jacobson, Lemma ② is implied by the following:

Today: Move toward

Lemma (2') Let S be fin.-gen. R-alg.
Then R Jacobson $\Rightarrow$ S Jacobson.

First, an exercise:

Exer If R Jacobson, so is R/I for any I.

Pf: Fix $\mathfrak{p} \subset R/I$ and let $\pi : R \to R/I$ be quotient map. Then $\pi^{-1}(\mathfrak{p})$ is prime, hence

$$\pi^{-1}(\mathfrak{p}) = \bigcap_{\substack{\pi^{-1}\mathfrak{p} \subset \mathfrak{m} \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m}$$

since R is Jacobson. Note π^{-1} induces bijection $\{\mathfrak{n} \text{ maximal in } R/I\} \xrightarrow{\cong} \{\mathfrak{m} \supset I\}$.

$$\pi^{-1}(\mathfrak{p}) \supset \bigcap_{\pi^{-1}(\mathfrak{p}) \subset \mathfrak{m}} \mathfrak{m} = \bigcap_{\mathfrak{p} \subset \mathfrak{n}} \pi^{-1}(\mathfrak{n})$$

$$\Rightarrow \underbrace{\pi(\pi^{-1}(\mathfrak{p}))}_{= \mathfrak{p}} \supset \pi\Big(\bigcap_{\mathfrak{p} \subset \mathfrak{n}} \pi^{-1}(\mathfrak{n})\Big) = \underbrace{\pi\left(\left\{x \;\middle|\; \begin{array}{l}\pi(x) \in \mathfrak{n} \\ \forall\, \mathfrak{n} \supset \mathfrak{p} \text{ maximal}\end{array}\right\}\right)}_{= \bigcap_{\mathfrak{p} \subset \mathfrak{n}} \pi^{-1}(\mathfrak{n})} \quad \text{since } \pi \text{ surjective}$$

Since $\mathfrak{p} \subset \bigcap_{\mathfrak{n} \supset \mathfrak{p}} \mathfrak{n}$ is obvious, we conclude $\mathfrak{p} = \bigcap_{\mathfrak{p} \subset \mathfrak{n}} \mathfrak{n}$. //

Also:

The interaction of complements and subspaces:

In homework, you proved:

$$L \to M \to N \text{ exact} \implies U^{-1}L \to U^{-1}M \to U^{-1}N \text{ exact.}$$

In particular, let's fix an ideal $I \subset R$ and consider the exact sequence

$$0 \to I \to R \to R/I \to 0$$

of R-modules. We have an exact sequence

$$0 \to U^{-1}I \to U^{-1}R \to U^{-1}(R/I) \to 0,$$

from which we conclude: The natural map

$$U^{-1}(R/I) \leftarrow U^{-1}R / U^{-1}I \qquad \text{(from the univ prop of quotients)}$$

is an $\cong$ of $U^{-1}R$-modules.

<u>Exer</u> This isom is a ring isomorphism.

<u>Pf</u> The map sends $\frac{x}{u} + U^{-1}I$ to $\frac{x+I}{u}$; ie, $[\frac{x}{u}] \mapsto \frac{[x]}{u}$, where $[\frac{x}{u}] = [\frac{x}{u}]_{U^{-1}I}$, $[x] = [x]_I$. Then

$$[\tfrac{x}{u}][\tfrac{y}{v}] = [\tfrac{xy}{uv}] \mapsto \frac{[xy]}{uv} = \frac{[x][y]}{uv} = \frac{[x]}{u} \cdot \frac{[y]}{v}.$$

Meanwhile $[\frac{1}{1}] \mapsto \frac{[1]}{1}$, so mult. id. is respected. //

We refine our knowledge of how localization transfers ideals:

Prop. Let $U \subset R$ be mult-closed, $R \xrightarrow{i} U^{-1}R$ the usual ring map. Then

(1) $\{\text{ideals } I \subset U^{-1}R\} \xrightarrow{i^{-1}} \{\text{ideals } J \subset R\}$ is an injection

(2) The image $i^{-1}\{\text{proper ideals}\} \subset \{J \text{ st } J \cap U = \phi\}$. (Not usually equality.)

(2') The image $i^{-1}(\mathrm{Spec}(U^{-1}R)) = \{J \subset R \text{ st } J \text{ prime}, J \cap U = \phi\}$

Pf We already did (1), but for completeness: Given $I \subset U^{-1}R$, $J := I \cap R$ is the set $\{\frac{x}{1} \mid x \in R, \frac{x}{1} \in I\}$. Moreover, I is an ideal, so $J \subset I \Rightarrow U^{-1}J \subset I$, where

$$U^{-1}J := \{\tfrac{x}{u} \mid x \in J, u \in U\} \subset U^{-1}R.$$

But $\frac{y}{b} \in I \Rightarrow b \cdot y/b \in J \Rightarrow \frac{1}{b}(b \cdot y/b) = y/b \in U^{-1}J$, so $I \subset U^{-1}J$.

$$\Rightarrow I = U^{-1}(I \cap R)$$

$$\Rightarrow I \text{ is determined uniquely by } I \cap R$$

As for (2'), suppose $p \subset U^{-1}R$ is prime. We know i^{-1} preserves prime ideals; moreover,
$p \neq U^{-1}R \Rightarrow U \cap (p \cap R) = \phi$. (If $u \in p \cap R$, then $\frac{u}{u} = 1 \in p$.)
this proves (2).
So p prime $\Rightarrow p \cap R$ is a prime st $p \cap R \cap U = \phi$.

In the other direction, if $p \cap R = q$ is prime and $q \cap U = \phi$, let $p = U^{-1}q = \{\frac{x}{u} \mid x \in q\}$.
Then p is prime because

$$\frac{a}{u} \cdot \frac{b}{v} \in p \Rightarrow \frac{a}{u} \cdot \frac{b}{v} = \frac{x}{w}, \ x \in q, w \in U$$

$$\Rightarrow \exists\, u_0 \text{ st } u_0(wab - xuv) = 0$$

$$\Rightarrow u_0 wab \in q$$

$$\Rightarrow ab \in q \text{ since } U \cap q = \phi$$

$$\Rightarrow a \in q \text{ or } b \in q \text{ since } q \text{ prime}$$

$$\Rightarrow \frac{a}{u} \in U^{-1}q \text{ or } \frac{b}{v} \in U^{-1}q. \quad //$$

What does this all mean geometrically? Let $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(R/I)$

$$R \longrightarrow U^{-1}R \quad \longleftrightarrow \quad \text{inclusion of the (open) subspace}$$
$$X \setminus V(U) \hookrightarrow X$$

$$R \longrightarrow R/I \quad \longleftrightarrow \quad \text{closed subspace inclusion}$$
$$Y \hookrightarrow X$$

$$R/I \to U^{-1}(R/I) \quad \longleftrightarrow \quad \text{inclusion of open subspace}$$
$$Y \setminus V(U) \hookrightarrow Y$$

$$U^{-1}R \longrightarrow U^{-1}R/U^{-1}I \quad \longleftrightarrow \quad \text{inclusion of closed subspace}$$
$$\mathrm{Spec}(U^{-1}R/U^{-1}I) \hookrightarrow X \setminus V(U)$$

Note $U^{-1}R/U^{-1}I$ represents subspace of X_U along which I vanishes — ie, it's $(X \setminus V(U)) \cap Y$

The isomorphism $U^{-1}R/U^{-1}I \xrightarrow{\cong} U^{-1}(R/I)$ expresses

$$(X \setminus V(U)) \cap Y \cong Y \setminus V(U),$$

which is geometrically intuitive:

$$(X \setminus V(U)) \cap Y = (X \cap Y) \setminus (V(U) \cap Y)$$
$$= Y \setminus (V(U) \cap Y)$$
$$=: Y \setminus V(U) \quad \text{(lazy notation)} \quad //$$

Here's a lemma for the lemma:

$R/p[f^{-1}] = U^{-1}(R/p)$ where $U = \{f^k \mid k \geq 1\}$

Lemma TFAE:

(1) R is Jacobson

(2) $\forall\, p \subset R$ prime, if $\exists\, f \in R/p$, $f \neq 0$, s.t. $R/p[f^{-1}]$ is a field, then R/p is a field.

Pf: (1) $\Rightarrow$ (2) By exercise, R/p is Jacobson. Since R/p is an integral domain, (0) is prime, hence $\bigcap_{m \text{ maximal}} m = (0) \subset R/p$. We now claim (0) is a (the i.e., only) maximal ideal of R/p.

Recall from Prop'n that "$r \subset R/p$ prime, $r \cap \{f^k\} = \phi$" $\Leftrightarrow$ "$U^{-1}r \subset R/p[f^{-1}]$ is prime"
But $R/p[f^{-1}]$ is a field, so $U^{-1}r = (0)$ in $R/p[f^{-1}]$; since R/p has no zero divisors, $\frac{r}{f^k} = \frac{0}{1} \Leftrightarrow r = (0)$. Thus, if r is any non-zero prime ideal, r must contain f.

If any $r \neq (0)$ maximal exists, then, $f \in \bigcap_{m \text{ maximal}} r$, which contradicts R/p being integral and Jacobson. We conclude (0) is a maximal ideal of R/p, hence R/p is a field.

(2) $\Rightarrow$ (1) If $\exists\, p$ st $p \not\supset \bigcap_{m \supset p} m$, fix f st $f \notin p$, $f \in \bigcap_{m \supset p} m$. Then there is a prime ideal q, maximal among those ideals which contain p but do not contain f. Note q is NOT maximal (because $f \notin q$ and $p \subset q$), by cnstrxn

Then $f \in R/q$ (abusing notation) is non-zero; meanwhile, $R/q[f^{-1}] \cong R[f^{-1}]/(q)$ (other exercise) is a field because $(q) = U^{-1}q$ is maximal in $U^{-1}R = R[f^{-1}]$. (by Prop'n (2))

(2) $\Rightarrow$ R/q is a field, $\rightarrow\leftarrow$ since q not maximal. //

Rmk As usual, let's try to interpret this lemma geometrically.

(1) Recall R Jacobson $\iff$ $\forall\, p$ prime, $p = \bigcap_{\substack{p \subset m \\ m \text{ maximal}}} m$.

i.e., "any irreducible, non-infinitesimal subset is a union of its points."

(2) Note that condition (2) implies $(R/p)[f^{-1}] \cong R/p$. Geometrically:

- Look at an irreducible in $\operatorname{Spec}(R)$ (called $\operatorname{Spec}(R/p)$).
- Suppose that the complement of a single function f is just a point.
- Then that irreducible was a point to begin with, and f was nowhere-vanishing.

($\operatorname{Spec}((R/p)[f^{-1}])$ is like $\operatorname{Spec}(R/p) \setminus V(f)$. That $R/p\,[f^{-1}]$ is a field means this complement is a "point".)

Intuitively, if X is irreducible, there's no closed subset $K \subset X$ st $K \amalg pt = X$. So $K = V(f)$ must be empty, and $pt = X$.

Next time: Prove Lemma (2).