<u>Homework</u>: Let $\mathbb{D} := \mathbb{C}[x]/(x^2)$. $\mathbb{D}$ is for "dual," as $\mathbb{D}$ is sometimes called the "dual numbers."

Fix a ring homomorphism

$$R \longrightarrow \mathbb{C}$$

which we think of as a point of Spec(R). (The kernel picks out a maximal ideal of R.) Call the point x.

Then a tangent vector at x is a commutative diagram

$$\begin{tikzcd} R \arrow[r, "v"] \arrow[dr, "x"'] & \mathbb{D} \arrow[d] \\ & \mathbb{C} \end{tikzcd}$$

where $\mathbb{D} \to \mathbb{C}$ is the natural quotient, $[a+bx] \longmapsto a$.

What explains the <u>addition</u> structure of $\hom_x(R, \mathbb{D}) = \left\{ \begin{tikzcd} R \arrow[r, "v"] \arrow[dr, "x"'] & \mathbb{D} \arrow[d] \\ & \mathbb{C} \end{tikzcd} \right\}$?

Given two maps v_1, v_2, we have

$$\begin{tikzcd} R \arrow[r, "v_1"] \arrow[d, "v_2"'] & \mathbb{D} \arrow[d] \\ \mathbb{D} \arrow[r] & \mathbb{C} \end{tikzcd}$$

hence by univ. prop. of fiber products (of rings), we have

$$
\begin{tikzcd}
R \arrow[dr, "\exists!"] \arrow[drr, bend left, "v_1"] \arrow[ddr, bend right, "v_2"'] & & \\
& \mathbb{D} \times_{\mathbb{C}} \mathbb{D} \arrow[r] \arrow[d] & \mathbb{D} \arrow[d] \\
& \mathbb{D} \arrow[r] & \mathbb{C}
\end{tikzcd}
$$

where you can check $\mathbb{D} \times_{\mathbb{C}} \mathbb{D} \cong \{ a + b_1 x_1 + b_2 x_2 , \quad a, b_1, b_2 \in \mathbb{C} \}$

with multiplication

$$(a + b_1 x_1 + b_2 x_2)(a' + b_1' x_1' + b_2' x_2)$$
$$\|$$
$$aa' + (ab_1' + a'b_1)x_1 + (ab_2' + a'b_2)x_2 .$$

We have a ring homomorphism

$$\mathbb{D} \times_{\mathbb{C}} \mathbb{D} \longrightarrow \mathbb{D}$$

$$(a, b_1, b_2) \longmapsto a + (b_1 + b_2)x .$$

Check: $(a + (b_1 + b_2)x)(a' + (b_1' + b_2')x) = aa' + (ab_1' + ab_2' + a'b_1 + a'b_2)x$

That is, if $\mathrm{Rings}_{/\mathbb{C}} =: \mathcal{C}$ is the category of rings equipped w/ a map to $\mathbb{C}$,

$\mathcal{C}$ has a monoidal structure (aka direct product) called $(R \to \mathbb{C}, S \to \mathbb{C}) \longmapsto (R \times_{\mathbb{C}} S \to \mathbb{C})$

Then $\mathbb{D} \to \mathbb{C}$ is an "abelian group object" in $\mathcal{C}$:

$$\mathbb{D} \times_{\mathbb{C}} \mathbb{D} \longrightarrow \mathbb{D} \qquad \begin{array}{c} \mathbb{C} \longrightarrow \mathbb{D} \\ a \longmapsto a \end{array} \qquad \begin{array}{c} \mathbb{D} \longrightarrow \mathbb{D} \\ (a,b) \longmapsto (a,-b) \end{array}$$

"group operation" "unit" "inverse"

where inverse means:

$$(a,b) \longmapsto (a,b,-b) \longmapsto a$$

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "\mathrm{id} \times_{\mathbb{C}} \mathrm{inv}"] \arrow[dr] & \mathbb{D} \times_{\mathbb{C}} \mathbb{D} \arrow[r, "m"] \arrow[d, phantom, "\circlearrowleft"] & \mathbb{D} \\
& \mathbb{C} \arrow[ur, "\mathrm{unit}"'] &
\end{tikzcd}$$

commutes.

just as w/ groups:

$$g \longmapsto (g, g^{-1})$$

$$\begin{tikzcd}
G \arrow[r] \arrow[dr] & G \times G \arrow[r, "m"] & G \\
& * \arrow[ur] &
\end{tikzcd}$$

This explains <u>additive</u> structure of $\hom_X(R, \mathbb{D})$.

Continuing curves: Last time:

Thm R Noetherian, $\dim R = 1$, R domain.

(1) R has unique fact of ideals: $\forall I$, $I = p_1^{n_1} \cdots p_k^{n_k}$ uniquely.

(2) R is locally a PID.

(3) R is integrally closed.

Defn Let R be Noeth, dim 1 ring, domain satisfying any (hence all) of (1)~(3). We say R is a Dedekind domain.

Rmk A Dedekind domain is like a non-singular curve!

Recall:

Thm $\mathbb{k}$ alg closed, $f \in \mathbb{k}[x,y]$. Fix $a \in Z_f = \{(x_0,y_0) \mid f(x_0,y_0)=0\} \subset \mathbb{k}^2$.
Then Z_f is NOT singular @ a iff
$$m \subset (Z_f)_m$$
is principal.

In homework, you'll prove:

Prop R a PID $\iff$ R domain, and every prime ideal is principal.

Note that since $\dim\big(\underset{\underset{\mathcal{C}_f}{\parallel}}{\overline{\mathbb{k}}[x,y]/(f)}\big) \leq 1$, we know $\dim\big((\mathcal{C}_f)_m\big) \leq 1$

because $(\mathcal{C}_f)_m$ is a localization of $\mathcal{C}_f$. (Primes in $\mathcal{C}_f$ avoiding m $\longleftrightarrow$ Primes in $(\mathcal{C}_f)_m$.)

On the other hand, f irreducible $\Rightarrow (f)$ prime
$\Rightarrow \mathcal{C}_f$ int. domain
$\Rightarrow (\mathcal{C}_f)_m$ domain,

so we have a chain $(0) \subset m$ in $(\mathcal{C}_f)_m$, showing $(\mathcal{C}_f)_m$ has dim 1. Since $(\mathcal{C}_f)_m$ is local, its only prime ideal is hence m.

Thus, by HW prop, $(\mathcal{C}_f)_m$ is a PID.

<u>Cor</u> Z_f is nonsingular iff $\mathcal{C}_f$ is locally a PID.

<u>Pf</u> (That Z_f not singular @ $a \Rightarrow m_a \subset (\mathcal{C}_f)_{m_a}$ principal. You'll prove converse in homework.)

By applying ring isomorphism $\overline{\mathbb{k}}[x,y] \longrightarrow \overline{\mathbb{k}}[x,y]$ $\quad x \longmapsto x - x_0,\ y \longmapsto y - y_0$

(geometry: translation) we assume a is the origin: $a = (0,0)$.

WLOG, assume $\left.\frac{\partial f}{\partial y}\right|_{(0,0)} \neq 0$. These two assumptions mean

f is of the form

- f has no constant term ($f(0,0)=0$)
- If
$$f = \sum_{i \geq 0}^{n} g_i(x) y^i, \qquad g_i(x) \in \overline{k}[x]$$
then $g_1(x)$ has a constant term, ε.

(else $\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = g_1(0) = 0.$)

We hence conclude

$$f = \varepsilon y + \underbrace{g(x)}_{\substack{\parallel \\ g_0(x):\ \text{no constant term.}}} + \overbrace{\sum_{i \geq 1} h_i(x) y^i}^{\text{evaluates to } 0 \text{ @ origin.}}, \qquad \varepsilon \neq 0 \in \overline{k}$$

Pass to G_f:

$$0 = g(x) + y\left(\varepsilon + \sum h_i y^i\right)$$

$\Longrightarrow$

$$g(x) = y(\varepsilon + h) \qquad \text{(being cavalier w/ signs.)}$$

Now, LHS has no constant term, so $g(x) \in (x)$. This means

$$(x) \ni y(\varepsilon + h).$$

Pass to $(G_f)_m$:

by assumptions, $h(x,y)$ is a polynomial such that $h(0,0)=0$.
So $h \in m \subset R$; hence $h \in m \subset (C_f)_m$.

<u>Exer</u> R local ring. ε unit, $h \in m$.
Show $\varepsilon + h$ is a unit in R.

<u>Soln</u>: By Zorn's lemma, any non-unit is contained in a maximal ideal m. (Take poset to be ideals w/ the non-unit, but which don't contain 1.) $\varepsilon + h$ not unit $\Rightarrow \varepsilon + h \in m \Rightarrow \varepsilon \in m \rightarrow\leftarrow$. //

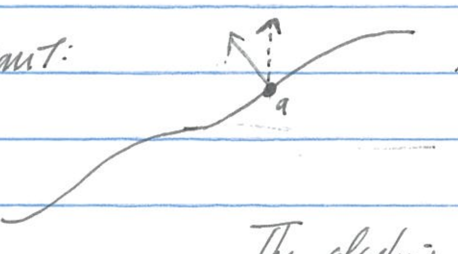
Since $\varepsilon + h$ is a unit,

$$(x) \ni y(\varepsilon + h) \implies (x) \ni y.$$

Since $m = (x,y) \subset R$, we know $m = (x,y)$ in $(C_f)_m$.
Hence we conclude

$$m = (x) \text{ in } (C_f)_m. \quad //$$

Geometry: $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is gradient. Assuming $\frac{\partial f}{\partial y} \neq 0$ means gradient has a y component:



. By implicit fn thm, we expect y to locally be expressed as a fn of x.
The algebraic geometry analogue is that $y \in (x)$ in $(C_f)_m$.