Curves:

A "curve" is something one-dimensional.

(a curve)

We've seen some one-dimensional things already:

(1) $\Bbbk[[x]]$
(2) $\Bbbk[x]$
(3) $\mathbb{Z}$

One way to make new curves: add a dimension, then solve an equation.

$\underline{\text{Ex}}^{①}$ Let $R = \Bbbk[x,y] \cong \Bbbk[x][y]$ (adding "y direction" to $\Bbbk[x]$)
We know $\dim R = 2$. Choose a function $f(x,y)$ to solve (i.e., an element of R).
Then we might expect $R/(f)$ to be a curve.

$\underline{\text{Ex}}^{②}$ Let $R = \mathbb{Z}[x]$ (adding "x direction" to $\mathbb{Z}$). Given $f \in \mathbb{Z}[x]$, we might hope for / think of $R/(f)$ as a curve.

We'll begin w/ understanding Example① better, where we have geometric questions. (E.g., is the curve smooth?) Once we translate some geometry into algebra, we'll explore more about Example②; these explorations will lead us into Galois theory.

Plane curves Throughout, we assume f is irreducible!

Let $\Bbbk$ be a field, and $f \in \Bbbk[x,y]$. Let $Z_f := V(f) = \{(x,y) \in \Bbbk^2 \mid f(x,y) = 0\}$

Defn Suppose $a = (x_0, y_0) \in \Bbbk^2$, and $f(a) = 0$.
We say Z_f is *singular* at a if

$$\frac{\partial f}{\partial x}\Big|_a = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}\Big|_a = 0.$$

Rmk What do we mean by a derivative over an arbitrary field $\Bbbk$?
We define formally/algebraically, without "limits." If $f \in R[x]$, and

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n = \sum_{i=0}^{n} a_i x^i$$

we define

$$\frac{df}{dx} := 0 + a_1 + 2a_2 x + \cdots + n a_n x^{n-1} = \sum_{i=0}^{n} i \cdot a_i x^{i-1}.$$

As is custom, when $f \in R[x,y]$, $\frac{\partial f}{\partial y}$ refers to the derivative when f is thought of as an element of $R'[y]$, $R' = R[x]$.

Rmk If $\operatorname{char} \Bbbk = p > 0$, one can have polynomials which look non-constant, but have zero derivative. For example, if $\Bbbk = \mathbb{Z}/p\mathbb{Z}$, and

$$f(x) = a_p x^p + a_0 \qquad \left(\text{more genlly, } f(x) = \sum_{k \geq 0}^{n} a_{kp} x^{kp}\right)$$

then $\frac{df}{dx} = p \cdot a_p x^{p-1} = 0$.

Defn Let $f \in \mathbb{k}[x,y]$. If, $\forall a \in Z_f$ we have that f is not singular at a, we say f is a non-singular curve.

Ex Let $f = y - x^2$. Then $\frac{\partial f}{\partial x} = -2x$, $\frac{\partial f}{\partial y} = 1$.

Since $\frac{\partial f}{\partial y}$ is never zero, Z_f is non-singular.

Ex Let $f = y^2 - x^3$. Then $\partial f/\partial x = -3x^2$, $\partial f/\partial y = 2y$. Since $a = (0,0) \in Z_f$, Z_f is singular @ $a = (0,0)$.

In principle, given f, we can compute on a case-by-case basis to see if Z_f is singular. But we become more powerful if we can tie "non-singularity" to broader principles.

Prop 8.1

① **Thm** Let $\mathbb{k}$ be algebraically closed, and $f \in \mathbb{k}[x,y]$ irreducible. Fix $a \in Z_f$ and let $m = m_a = (x - x_0, y - y_0)$ be the maximal ideal given by a. Then f is non-singular at a iff the local ring $(\mathbb{k}[x,y]/(f))_m$ is a PID.

By Nullstellensatz, we can say: f is non-singular iff $\forall m \subset \mathbb{k}[x,y]/(f)$ maximal, m is principle in the localization of $\mathbb{k}[x,y]/(f)$ at m.

Notation: Given f irreducible, we let $C_f := \mathbb{k}[x,y]/(f)$.

The theorem "confirms" that singularity is a *local* property; it suffices to check at each maximal ideal. However, there's also a global check:

Thm ② Let $f \in \overline{\mathbb{k}}[x,y]$ be irreducible. Then Z_f is non-singular if and only if C_f is integrally closed.

Recall: A ring R is called *integrally closed* if the integral closure of R inside $\mathrm{Quot}(R) = \mathrm{Frac}(R)$ is R itself.

Now let's abstract from $\overline{\mathbb{k}}$-algebras to rings in general. Note that any C_f is of dimension 1 and Noetherian.

Exer Prove C_f is dimension 1 when $f \neq 0$.

Sol'n: f irreducible $\Rightarrow (f)$ prime, so C_f a domain. Let $(0) = p_0 \subset \cdots \subset p_n \subset C_f$ be a chain of primes, and $\pi : \overline{\mathbb{k}}[x,y] \longrightarrow C_f$ the quotient map. Then

$$(f) = \pi^{-1}(p_0) \subset \cdots \subset \pi^{-1}(p_n)$$

is a chain of primes in $\overline{\mathbb{k}}[x,y]$, which has dimension *two*:

$$(0) \subset (f) \subset \pi^{-1}(p_1)$$

so C_f has $\dim \leq 1$. Now we need to show:

Lemma $\mathbb{k}$ any field, $f \in \mathbb{k}[x,y]$ irreducible. Then (f) is NOT maximal.

This lemma's not obvious!

Regardless, Thms ① and ② show that for Noetherian, dimension one algebras of the type $k[x,y]/(f)$, "non-singular" coincides with algebraic properties. In fact:

<u>Thm</u> Let R be Noetherian, dimension one. TFAE:
(1) R is integrally closed
(2) $\forall\ m \subset R$ maximal, R_m is a PID
(3) R admits unique factorization of ideals.

(Rmk: (1) and (3) are also true iff they're true locally.)

<u>Defn</u> A ring R has unique fact. of ideals if $\forall\ I \subset R$, $\exists$ primes $p_i \subset R$ st

$$I = \bigcap_{\text{finite}} p_i^{n_i}$$

and moreover, if $I = \bigcap_{\text{finite}} q_j^{m_j}$,

$\exists$ bijection $\{i\} \cong \{j\}$ st $p_i = q_j$, $n_i = m_j$.

<u>Defn</u> Any Noetherian, dimension-one R satisfying any of (1)~(3) is called a <u>Dedekind domain</u>.

<u>Rmk</u> The theorem above allows us to connect "non-singularity" (geometry) to factorization (algebra).

The main insight into geometry comes from the following:

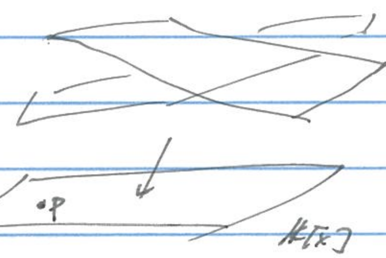
<u>Thm</u> Let A be a Dedekind domain, and let L be a finite extension of $\mathrm{Quot}(A) = \mathrm{Frac}(A)$. Then the integral closure B of $A \subset L$ is a Dedekind domain.

<u>Ex</u> $A = \Bbbk[x]$; choose $f \in \Bbbk[x,y] = A[y]$. Then consider

$$L := \mathrm{Frac}(A)[y]/(f)$$

Note $f = \sum a_{ij} x^i y^j = \sum g_j(x) y^j$ is a fin-deg polynomial over $\mathrm{Frac}(A)$, so L is a finite-degree extension of $\mathrm{Frac}(A)$. Let $B \subset L$ be the integral closure of A. Algebraically, we have the inclusion $A \hookrightarrow B$; geometrically, this induces a map

$$\begin{array}{c} \mathrm{Spec}\, B \\ \downarrow \\ \mathrm{Spec}\, A \end{array}$$



non-singular (since B Dedekind)

Let P be a maximal ideal of A. Then the induced ideal $P \cdot B \subset B$ can be shown to be proper..., and we have a unique factorization

$$PB = Q_1^{e_1} \cdots Q_k^{e_k}$$

by maximal ideals.

Exer R local ring w/ $m \subset R$ maximal.
If $u \in R$ is a unit and $g \in m$,
then $u+g$ is a unit in R_m.

Pf $u+g \notin m$, so $\frac{1}{u+g} \in R_m$. //

Pf (that Z_f non-singular @ $a \implies (C_f)_{m_a} \supset m_a$ is principal)

WLOG, assume $a=0$.
WLOG, assume $\frac{\partial f}{\partial y}\big|_{(0,0)} \neq 0$; set $u = \frac{\partial f}{\partial y}\big|_{(0,0)} \in \mathbb{k}$.
Since $f(0,0)=0$, have

$$f(x,y) = \sum_{i\geq 1}^{m} b_i x^i + \sum_{j=1}^{n} c_j y^j + \sum_{i,j\geq 1} a_{ij} x^i y^j.$$

By derivative condition, have:

$$\frac{\partial f}{\partial y}\Big|_{(0,0)} = c_1 = u.$$

In C_f, $[f]=0$, so

$$-\sum b_i x^i = y\left(u + \sum_{i\geq 1} g_i(x) y^i\right), \text{ in } C_f.$$

Note that $\sum g_i(x) y^i = g$ has no constant terms, so $g \in m$. By exercise, $u+g$ is a unit, so $y \in (x)$. Since $m = (x,y)$, we conclude $m = (x)$. //

Converse: Homework!

Prop. A ring. TFAE:

(1) A is a PID

(2) Every prime ideal of A is principal.

Pf. (1) $\Rightarrow$ (2) obvious.

(2) $\Rightarrow$ (1): Let $\mathcal{P} = \{I \subset A \mid I \text{ not principal}\}$. Claim: $(\mathcal{P}, \subseteq)$ satisfies Zorn's Lemma.

Pf of claim: Given $I_0 \subset I_1 \subset \cdots$, let $I = \bigcup_{n \geq 0} I_n$.
If $I = (a)$, then $a \in I$, so $\exists$ n st
$a \in I_n \Longrightarrow (a) \subset I_n. \Longrightarrow (a) = I_n \rightarrow\leftarrow$.

So let I be maximal. As usual, I will be prime. (Exercise.)
But assumption (2) says I can't be in $\mathcal{P}$! $\rightarrow\leftarrow$. Hence $\mathcal{P}$ is empty.