Monday, Nov 20, 2017

Recall:

Thm A Noetherian, dim 1.
TFAE:

(1) Unique fact of ideals
(2) Locally PID
(3) Integrally closed.

Def Any A satisfying (1)~(3) is called a **Dedekind domain**.

The main motivation: $\operatorname{Spec}(A)$ non-singular $\longleftrightarrow$ A Dedekind

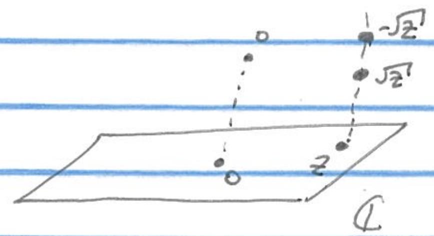
Some basic geometry of smooth curves.

Consider the function

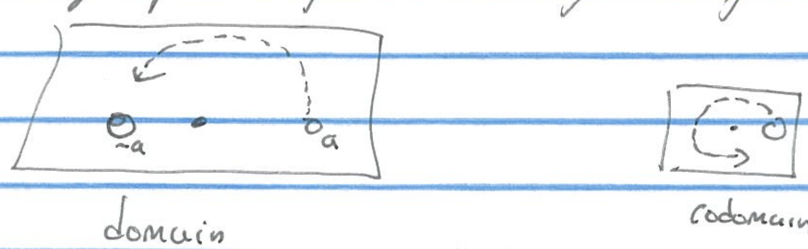
$$\mathbb{A}^1(\mathbb{C}) = \mathbb{C} \longrightarrow \mathbb{C} = \mathbb{A}^1(\mathbb{C}).$$
$$z \longmapsto z^n$$

<u>Ex</u> $(n=2)$

Over the codomain, the fibers of this map look like:

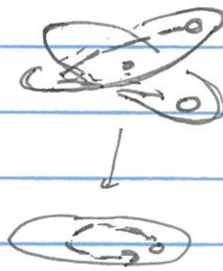


ie, fibers have cardinality n everywhere except at origin, where fiber is a single point. Another interesting bit of structure is <u>monodromy</u>:



where "lifting" a loop in codomain results in a path from a to $-a$ in the domain.

Or, here's another way to visualize:



$$\begin{array}{c} \mathbb{A}^1 \\ \downarrow \\ \mathbb{A}^1 \end{array}$$

This number n ($=2$ in example) seems important. Generically, it's the size of the fiber. But what about at the origin? Is there a way to detect it?

Let $m = (x, y) \in \mathrm{MaxSpec}(\mathbb{C}[x]) \longleftrightarrow$ origin of $\mathbb{A}^1$.

The map $z \longmapsto z^2$ is modelled by

$$\begin{array}{ccccc} \mathbb{C}[z] & \longleftarrow & \mathbb{C}[x,y]/(y^2-x) & \longleftarrow & \mathbb{C}[x] \\ & & f(x) & \longleftarrow\!\shortmid & f(x) \\ h(z) & \longleftarrow\!\shortmid & \underline{g(x,y)} & & \\ & & \shortparallel & & \\ & & \underline{g(y^2,y)} & & \\ & & \shortparallel & & \\ & & h(y) & & \end{array}$$

i.e.,

$$f(z^2) \longleftarrow\!\shortmid f(x).$$

(Pulling back a polynomial $\underset{\text{codomain}}{\mathbb{A}^1} \xrightarrow{f} \mathbb{C}$ means "substitute x w/ z^2.")

So we have

$$\begin{tikzcd} \mathbb{C}[x,y]/(y^2-x) \arrow[r, hookrightarrow] & L := \mathbb{C}(x)[y]/(y^2-x). \\ \mathbb{C}[x] \arrow[u] \arrow[r, hookrightarrow] & K := \mathbb{C}(x) \arrow[u] \end{tikzcd}$$

Where $K \subset L$ is a Galois extension of degree 2.

What does it mean to study the fiber of a map?
Given a point $a \in \operatorname{Spec} A$ ($\longleftrightarrow$ a maximal ideal $m \subset A$),
the fiber of a in $\operatorname{Spec} B$ is the ideal generated by m in B.

That is, viewing $m \subset A \subset B$, it makes sense to consider $mB \subset B$, the ideal generated by B.

Ex $\quad A = \mathbb{C}[x] \hookrightarrow \mathbb{C}[x,y]/(y^2-x) \cong \mathbb{C}[z]$

$$f(x) \longmapsto f(z^2).$$

If $m = (x-a)$, $a \neq 0$, then $mB = (z^2-a) = (z+\sqrt{a})(z-\sqrt{a})$ in B.
That is,

$$mB = M_1 M_2$$

for two maximal ideals of B. (Such a factorization exists when B is Dedekind!)

If $m = (x)$ $(a=0)$, then $mB = (z^2) = (z)^2$. So $mB = M^2$ for M maximal.

So the number of factors (w/ multiplicity) in the prime ideal factorization of mB seems to recover 2 $(=n)$.

More examples:

$A = \mathbb{Z}$
$B = \mathbb{Z}[i]$.

The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[i]$ is

$$\mathbb{Z} \hookrightarrow \mathbb{Z}[x] \twoheadrightarrow \mathbb{Z}[x]/(x^2+1) \cong \mathbb{Z}[i].$$

This gives rise to

$$\begin{array}{ccccc} B & = \mathbb{Z}[i] & \hookrightarrow & \mathbb{Q}[i] & = L \\ & \cup & & \cup & \\ A & = \mathbb{Z} & \hookrightarrow & \mathbb{Q} & = K \end{array}$$

via fraction fields. Clearly, $[L:K]=2$.

$m=(2)$: $\quad mB = (2) = (1+i)(1-i) = (i)(1+i)^2 = (1+i)^2 = M^2$.

$m=(3)$: $\quad mB=(3)$, which is prime. This doesn't recover "2". However, consider $B/(3)B$ vs $A/(3)$. Note we have

$$A/m \hookrightarrow B/M,$$ so we have a finite field extension.

$B/M = B/(3) \cong \mathbb{Z}[i]/(3) \cong \mathbb{Z}[x]/(3,x^2+1) \cong \mathbb{F}_3[x]/(x^2+1) \cong \mathbb{F}_9$.
$A/m = A/(3) \cong \mathbb{Z}/3\mathbb{Z} \cong \mathbb{F}_3$.

$[B/M : A/m] = 2$! Recover 2.

$m=(5)$: Then $mB = 5B = (5) = (1-2i)(1+2i) = M_1 M_2$, have <u>two</u> distinct primes in factorization.

It's hard to see from these examples, but here's the general pattern:

<u>Defn</u> Fix
$$\begin{array}{ccc} B & \hookrightarrow & L \\ \cup & & \cup \\ A & \hookrightarrow & K \end{array}$$
satisfied if L separable over K

where $[L:K] < \infty$, B fin. gen. as A-module and
- A, B Dedekind
- K, L are frac fields of A, B, respectively
- B = integral closure of A in L.

(Turns out any two implies the third.) Then for any $m \in \mathrm{MaxSpec}(A)$, write

$$mB = M_1^{e_1} \cdots M_n^{e_n} \quad \text{prime fact of ideals in } B$$

We say e_i is the <u>ramification degree</u> of M_i over m.

Since $A/m \hookrightarrow B/M_i$ is a field extension,

$$f_i := [B/M_i : A/m]$$

is called the <u>residual degree</u> of M_i over m.

$$\begin{array}{ccc} B & \hookrightarrow & L \\ \cup & & \cup \\ A & \hookrightarrow & K \end{array}$$

Thm $\forall$ $m \in \mathrm{MaxSpec}(A)$, as above,

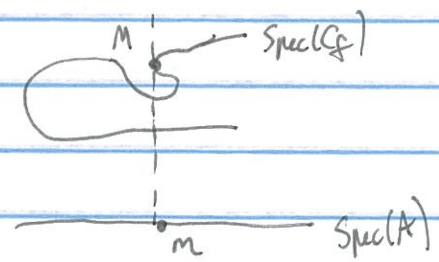
$$[L:K] = \sum_{M_i} e_i f_i .$$

Rmk We saw when $A = \mathbb{Z}$, $B = \mathbb{Z}[i]$ that to compute f_i, we needed to do computations mod p — ie, positive characteristic naturally appears!

Defn M is ramified over A (or over $m = M \cap A$) if
- $e > 1$, or
- $A/m \hookrightarrow B/M$ is not separable.

Thm A Dedekind, $f \in A[y]$ monic irreducible, and $M \in \mathrm{MaxSpec}(C_f)$. If C_f is a Dedekind domain, M is ramified over A iff $f'(y) \in M$. (I.e. if $f'(\alpha) = 0$ in C_f/M).

Rmk.



$$A \hookrightarrow A[y]/f \cong C_f$$

$$A/m \hookrightarrow C_f/M \ni y.$$

$$\cong A[y]/(f, m, M)$$

$$\cong A/m[y]/(f, M) \cong A/m[\alpha]/M \cong A/m[\alpha]$$

some root of f.

Now, Galois theory:

Thm If $K \subset L$ finite Galois,

$$\begin{array}{ccc} B & \hookrightarrow & K \\ \cup & & \cup \\ A & \hookrightarrow & L \end{array}$$

as before, then

(1) $\forall\ \sigma \in Gal(L/K)$, $\sigma(B) = B$.

(2) $\forall\ m \in MaxSpec(A)$, $\forall\ M, M'$ over m,
$\exists\ \sigma$ st $\sigma M = M'$.

(3) $e_M = e_{M'}$, $f_M = f_{M'}$ $\forall\ M, M'$ over m, so

$$mB = (M_1 \cdots M_k)^e$$

and $f \cdot e \cdot k = [L : K]$.

Ie, ramification looks the same at all ramification points above a given fiber. Note this restricts the # of ramification points above a given m.

Note: $\sigma(B) = B$ means $Gal(L/K)$ acts on B, (from left)

hence on $Spec(B)$ (from right)

Moreover,

$$(5) \quad A = B^G, \text{ and}$$

$$Spec(B)/G \cong Spec(B^G) = Spec(A).$$

We finish w/ a more concrete application.

Let $K = \mathbb{C}(t)$.

If $K \subset L$ is finite, let $L = K[\alpha]$. (charac $0 \Rightarrow L$ separable; apply prim elt thm) and set f to be irreducible polynomial of α:

$$f \in K(t)[x], \quad f(t,x) = a_n x^n + \cdots + a_0$$

where we can choose
- $a_i \in K[t]$ (by clearing denominators)
- $\gcd(a_n, \dots, a_0) = 1$,
- a_n monic in t.

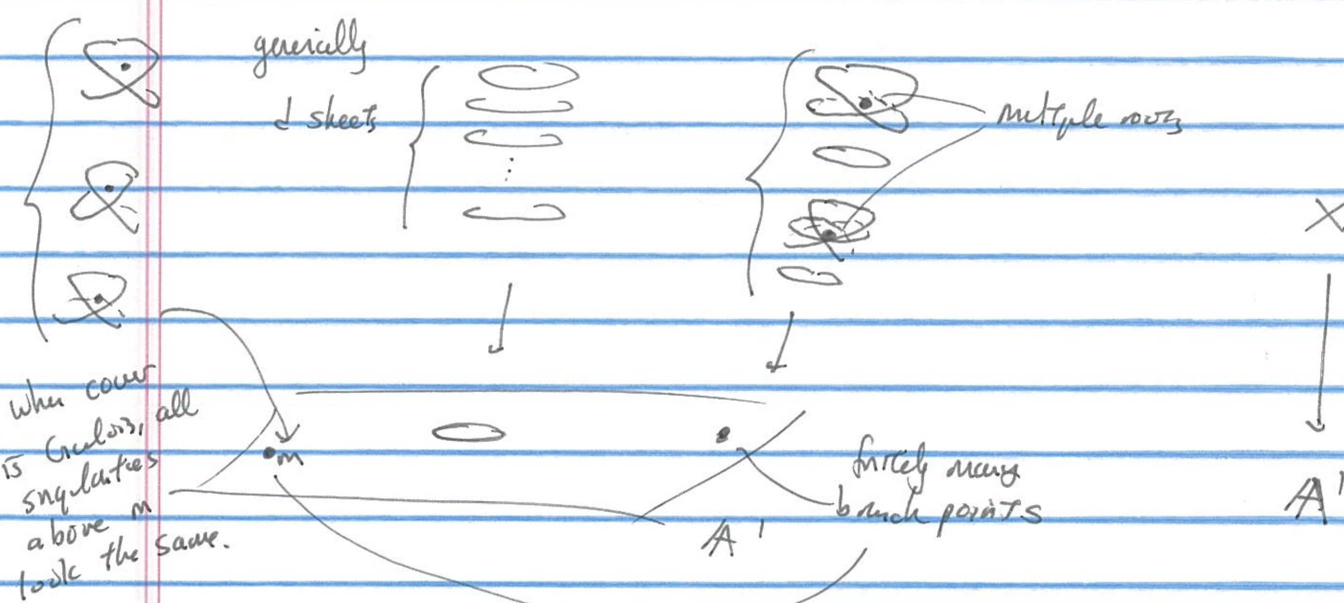
This f defines $Z_f \subset \mathbb{A}^2 = \{(t,x) \mid f(t,x) = 0\}$, w/ $Z_f \to \mathbb{A}^1$, $(t,x) \mapsto t$

having branch point $t_0 \in \mathbb{A}^1$ if $f(t_0, x)$ has repeated roots.

__Thm__ (Riemann Existence)

$$\left\{\begin{array}{c}\text{Isom classes of extensions}\\ \underset{\mathbb{C}(t)}{\overset{\parallel}{K}} \subset L,\ [L:K]=d\end{array}\right\} \cong \left\{\begin{array}{c}\text{isom classes}\\ \text{of } d\text{-sheeted}\\ \text{branched covers}\\ \text{of } \mathbb{A}^1\end{array}\right\}$$

$$\begin{tikzcd} L \arrow[rr, "\cong"] & & L' \\ & T \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "\cong"] \arrow[dr] & & X' \arrow[dl] \\ & \mathbb{A}^1 & \end{tikzcd}$$



__Rmk__ If $K \subset L$ Galois, over a given $m \in \mathbb{A}^1$, have that all pts ramified over p "look the same," by previous theorem.