Lie Groups and Lie algebras

Monday, Nov 27

When we studied representations of a finite group G, we had very little geometry to work with. At the same time, our constructions relied on a particular set-theoretic property of G: G is finite. This is what allowed us to do things like:

$$\begin{aligned} V &\longrightarrow V \\ x &\longmapsto \frac{1}{|G|}\sum_{g\in G} gx. \end{aligned}$$

(Take averages.)

Today, we'll combine groups with calculus.

Some calculus (aka smooth topology)

Defn Let $U \subset \mathbb{R}^n$ be an open set. A continuous $f: U \longrightarrow \mathbb{R}^m$ is C^k if the partial derivatives

$$\frac{\partial^{k'} f_i}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}, \quad \forall\ i=1,\dots,m$$

- exist $\forall\ k' \leq k$, $\forall\ j_1,\dots,j_n \geq 0$ w/ $j_1+\dots+j_n = k'$,
- and are continuous functions $U \longrightarrow \mathbb{R}$.

Defn f is smooth, or C^∞, if it's C^k $\forall\ k$.

Why smoothness?

(1) $k=1$ allows us to take derivatives. But then $\partial f/\partial x_i$ might be a crazy function — it may not even be continuous. So we demand derivatives should be "nice."

(2) But then we need to keep track of the regularity, or "how differentiable" each partial derivative is. In general, the derivatives of a k-times differentiable function are only $(k-1)$-times differentiable. "Smoothness" allows us to be lazy about keeping track of this regularity.

Rmk There are serious questions in both topology, geometry, and analysis to be asked about non-smooth functions, especially concerning the differences btwn $k=0$, $k=1$, $k=2$, and $k=\infty$

So the main tool we'll have now, which we didn't before, is the ability to take derivatives. (As many times as we want, though that won't matter.)

Defn A smooth submanifold of $\mathbb{R}^n$ is a subset

$$X \subset \mathbb{R}^n$$

satisfying two conditions:

(i) $\forall\ x \in X$, $\exists$ open $U_x \subset \mathbb{R}^n$ and C^∞ fxns

$$f_1, \dots, f_m : U_x \longrightarrow \mathbb{R}$$

such that

$$\overset{(a)}{x \in} U_x \cap X \overset{(b)}{=} \{(x_1, \dots, x_n) \mid f_i(x_1, \dots, x_n) = 0\ \forall i\}$$

and

(ii) The matrix

$$Df_x := \left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{i=1,\dots,m \\ j=1,\dots,n}} : T_x\mathbb{R}^n \longrightarrow T_0\mathbb{R}^m$$

is a surjection for all $x \in U_x \cap X$.

<u>Rmk</u> I've mentioned this condition in class as a way to define the <u>tangent space</u> of X at x as the kernel of Df_x.

<u>Rmk</u> The implicit fxn theorem implies that $\forall\ x \in X$, some open nbhd $U \subset X$ of x is isomorphic to an open subset of $\mathbb{R}^{n-m}$. Here, "isomorphic" means diffeomorphism.

<u>Defn</u> Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^N$

be smooth submanifolds. A function

$$f : X \longrightarrow Y$$

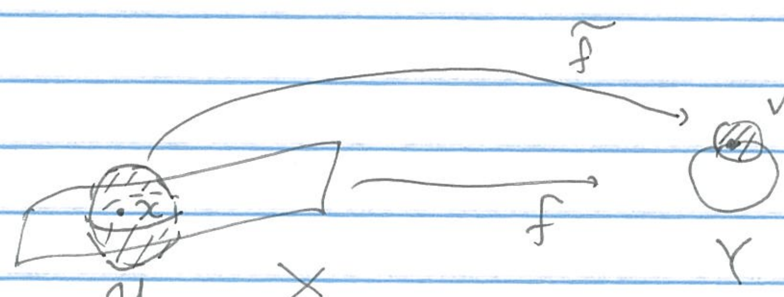
is called <u>smooth</u>, or <u>C^∞</u>, if $\forall\, x \in X$,
$\exists$ $U_x \subset \mathbb{R}^n$, $V_{f(x)} \subset \mathbb{R}^N$ open, and

$$\tilde{f} : U_x \longrightarrow V_{f(x)} \quad C^\infty$$

such that

$$\tilde{f}\big|_{X \cap U_x} = f\big|_{X \cap U_x}.$$

That is, f is smth if it <u>extends</u> <u>locally</u> to a smth fxn on some open neighborhood.



Defn A Lie Group is

the data of

$$(G, m)$$

where

- (G, m) is a group
- G is a smooth submfld of some $\mathbb{R}^n$, and
- The functions

$$m : G \times G \longrightarrow G \qquad (g, h) \longmapsto gh$$

$$(\ )^{-1} : G \longrightarrow G \qquad g \longmapsto g^{-1}$$

are both smooth

Rmk In general, a Lie group is "abstractly" a smth mfld; it need not come w/ a preferred embedding in $\mathbb{R}^n$. We are just taking a short path to the meat of the subject.

Ex $GL_n(\mathbb{R}) \cong \{(t, A) \in \mathbb{R}\times\mathbb{R}^{n^2} \mid t\cdot\det(A) = 1\}$

$$\subset \mathbb{R}^{1+n^2}.$$

This we saw algebraically, as motivation for localization. If we were smooth geometers or topologists, we would rather define:

$$GL_n(\mathbb{R}) := \{A \in \mathbb{R}^{n^2} \mid \det(A) \neq 0\}.$$

This is a submfld, locally defined by the empty collection of fxns, whose derivative we treat as the (surjective) zero map to the zero vector space.

Ex $GL_n(\mathbb{C}) := \{A \in \mathbb{C}^{n^2} \triangleq \mathbb{R}^{4n^2} \mid \det A \neq 0\}$ "general linear group"

Ex $U_n := \{A \in \mathbb{C}^{n^2} \mid A^*A = I\}$ where $(A^*)_{ij} = \overline{(A_{ji})}$.

"unitary group"

Ex $SU_n := \{A \in U_n \mid \det A = 1\}$ "special unitary group"

Ex $SL_n(\mathbb{R}) := \{A \in GL_n(\mathbb{R}) \mid \det A = 1\}$ "special linear group"

$SL_n(\mathbb{C}) := \{A \in GL_n(\mathbb{C}) \mid \det A = 1\}$

Ex $O_n(\mathbb{R}) := \{A \in GL_n(\mathbb{R}) \mid A^T A = I\}$

$SO_n(\mathbb{R}) := \{A \in O_n(\mathbb{R}) \mid \det A = I\}.$

Rmk $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$ are submflds b/c they're open subsets of $\mathbb{R}^{n^2}$, $\mathbb{C}^{n^2}$. Others require some checking, but we omit the verifications.

Rmk Matrix multiplication:

$$(A\cdot B)_{ik} = \sum A_{ij} B_{jk}$$

is polynomial in the $2\cdot n^2$ entries of (A,B), so is smth. (Polynomials have derivatives of all orders.) Moreover, Cramer's rule gives a polynomial formula for inverses, so

$$A \longmapsto A^{-1}$$

is also smooth. Since all groups above are submanifolds of GL_n ($\mathbb{R}$ or $\mathbb{C}$), the proof that m and $(\ )^{-1}$ are C^∞ for GL_n means they're C^∞ for each of the groups defined in the previous page.

Calculus and Groups

Defn Let G, H be Lie groups. A Lie group homomorphism is a smooth fxn

$$\phi: G \to H$$

which is also a gp homomorphism.

At this point, we observe:

(i) $\phi: G \to H$ sends $e_G \mapsto e_H$,

(ii) ϕ C^∞ $\Rightarrow$ we can take derivative

$$\Rightarrow D\phi_{e_G} : T_{e_G} G \longrightarrow T_{e_H} H.$$

Key idea

The entire theory of Lie groups rests on the observation that the derivative at the identity,

$$D\phi_{e_G}$$

captures everything about ϕ itself.

Rmk (1) Why? Let $U \ni e$ be an open nhd of the identity $e \in G$. Let

$$U_N := \{ x_1 \cdots x_N \mid x_i \in U \}$$

be the set of N-fold products. For G connected, one can show:

(1) $\displaystyle\bigcup_{N \geq 1} U_N = G.$ (Exercise for reader)

So local behavior near e captures global behavior.

Rmk ② But knowing derivative of a fxn ϕ at a single point cannot tell you ϕ.

(For example, $f=1$ and $f=1+x$ have same derivative at $x=0$.)

We must have a way of passing from "the derivative at a point" to "local" data (ie, data on an entire open set, not just a point.)

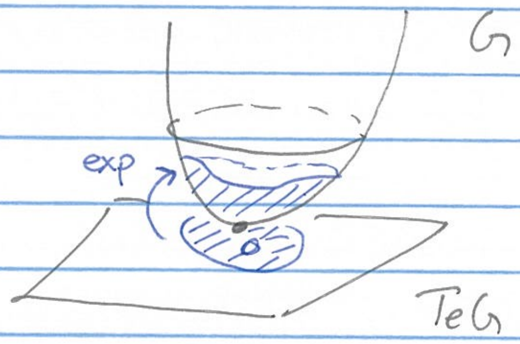
(2) Thm $\exists$ a smooth map

$$\exp : T_eG \longrightarrow G, \quad 0 \longmapsto e$$

called the exponential map (of G) which is a diffeomorphism on some open nbd of $0 \in T_eG$. Moreover,

$$\begin{tikzcd} G \arrow[r, "\phi"] & H \\ T_eG \arrow[u, "\exp"] \arrow[r, "D\phi_e"'] & T_eH \arrow[u, "\exp"'] \end{tikzcd}$$

commutes for any Lie gp homom ϕ.



exp is what gets us this *local* (not just "derivative at a point") data.

Rmk [3] But does any *linear* map $T_eG \to T_eH$ arise as the derivative of some Lie grp homomorphism? We want to know more about $D\phi_e$ now. In fact, since (1) and Remark [1] relies on the group structure of G, we also need to know how exp plays w/ group structure.

This is where *Lie algebras* come in. More on them soon. First, the big picture:

Thm T_eG has a Lie algebra structure.
Moreover, $D\phi_e$ is a Lie algebra homomorphism (if ϕ is a Lie grp homomorphism), and if

$$(D\phi_1)_e = (D\phi_2)_e$$

for ϕ_1, ϕ_2 Lie gp homoms, then

$$\phi_1 = \phi_2.$$

Defn We let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras associated to G, H.

Thus, a map of Lie algebras uniquely captures Lie gp homoms.

But does every Lie gp homom arise from a Lie alg homom?

Thm If G is simply-connected, any Lie algebra map

$$T_eG \cong \mathfrak{g} \longrightarrow \mathfrak{h} \cong T_eH$$

uniquely lifts to a Lie gp homom

$$\phi : G \longrightarrow H$$

s.t. $D\phi_e$ agrees w/ the Lie alg map.

Rmk Moreover, for G NOT simply-connected, one can form its universal cover $\widetilde{G}$, which will again be a Lie group, and for any Lie alg. map

$$T_eG \longrightarrow T_eH,$$

we can study the associated $\widetilde{G} \xrightarrow{\widetilde{\Phi}} H$. If $\mathrm{Ker}(\widetilde{\Phi})$ is large enough, $\widetilde{\Phi}$ descends to $\Phi : G \to H$.

Take-away: Information about Lie groups is completely captured by information about Lie algebras (up to π_1; ie, up to a topological obstruction of being simply connected)

Bonus Representation theory of a Lie group G is the special case of studying $G \xrightarrow{\phi} H$ when $H = GL_n(\mathbb{C})$.

We further have:

Thm: A representation $\Pi: G \to GL_n(\mathbb{C})$ is

- irreducible iff $\pi: \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$ is,

and two reps $\Pi_1, \Pi_2: G \to GL_n(\mathbb{C})$ are

- isomorphic iff π_1, π_2 are.

Lie Algebras]

Now let's study Lie algebras.

<u>Defn</u> Let $\mathbb{k}$ be a field of charac. 0.
A <u>Lie algebra</u> (over $\mathbb{k}$) is the data of:

- V a vector space over $\mathbb{k}$
- A $\mathbb{k}$-bilinear map

$$V \otimes_{\mathbb{k}} V \longrightarrow V, \quad x \otimes y \longmapsto [x, y]$$

called "the bracket,"

s.t.

(1) $[\ ,\ ]$ is anti-symmetric:

$$[x, y] = -[y, x]$$

[,] is a <u>derivation</u> for itself.

(2) $[\ ,\ ]$ satisfies the <u>Jacobi identity</u>:

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

<u>Rmk</u> No "unit". The "algebra" is an old term connoting some bilinear operation. Try <u>not</u> to fit this into some old notion of associativity or any such thing. Lie algebras are a <u>new</u> kind of algebraic structure, intimately tied to the infinitesimal.

Defn A *Lie algebra homomorphism* is a $\mathbb{k}$-linear map

$$f: V \longrightarrow W$$

such that

$$f([x,y]) = [fx, fy].$$

Ex Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) := M_{n\times n}(\mathbb{C})$. Define

$$[A,B] := AB - BA$$ ← usual matrix multiplication.

Exer: Check Jacobi identity.

- $[A,[B,C]] = A(BC-CB) - (BC-CB)A$
  $= ABC - ACB - BCA + CBA$
- $[[A,B],C] = (AB-BA)C - C(AB-BA) = ABC - BAC - CAB + CBA$
- $[B,[A,C]] = B(AC-CA) - (AC-CA)B = BAC - BCA - ACB + CAB$

Likewise for $\mathfrak{gl}_n(\mathbb{R})$, where $\mathbb{R} = \mathbb{k}$. Define:

$\mathfrak{sl}_n(\mathbb{k}) = \{A \in \mathfrak{gl}_n(\mathbb{k}) \mid \mathrm{trace}(A) = 0\} \subset \mathfrak{gl}_n(\mathbb{k})$

$\mathfrak{o}_n := \{A \in \mathfrak{gl}_n(\mathbb{R}) \mid A = -A^T\}$

$\mathfrak{so}_n := \{A \in \mathfrak{o}_n \mid \mathrm{trace}(A) = 0\}$

$\mathfrak{u}_n := \{A \in \mathfrak{gl}_n(\mathbb{C}) \mid A = -A^*\}$

$\mathfrak{su}_n := \{A \in \mathfrak{u}_n \mid \mathrm{trace}(A) = 0\}$

$\Big\}$ ⚠ $\mathbb{k} = \mathbb{R}$.

Home exercise: These are Lie subalgebras of $\mathfrak{gl}_n$.

<u>Defn</u> For any vec space V over $\mathbb{k}$, define

$$\mathfrak{gl}(V) := \mathrm{End}_{\mathbb{k}}(V)$$

w/ Lie bracket

$$[A,B] := AB - BA.$$

<u>Defn</u> Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ or $\mathbb{C}$.

A (complex) Lie algebra representation is a Lie alg. homom

$$\mathfrak{g} \longrightarrow \mathfrak{gl}(V), \quad \mathbb{R} \text{ or } \mathbb{C}\text{-linear}$$

for some V over $\mathbb{C}$.

<u>Rmk</u> If $\mathfrak{g}/\mathbb{R}$, any $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ for $V/\mathbb{C}$ extends uniquely to

$$\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathfrak{gl}(V)$$

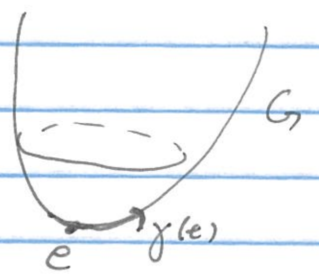
where $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ has obvious $\mathbb{C}$-linear Lie alg structure.

__Example__ Let $G = SL_2(\mathbb{C}) = \{ A \in M_{2\times 2}(\mathbb{C}) \mid \det A = 1 \}$.

Note we expect $\dim_{\mathbb{C}} G = 3$, since it's cut out from $M_{2\times 2}(\mathbb{C}) \cong \mathbb{C}^4$ by one eqn.

Since G is a group, it has an identity element $e \in G$. What is the tangent space of G at e?

(ie, what is $T_eG := \{$vectors tangent to G at $e\}$?)



Let $\gamma : (-\varepsilon, \varepsilon) \longrightarrow G$ be some curve
$0 \longmapsto e.$

By Taylor's theorem, we can write

$$\gamma(t) = \gamma(0) + \frac{d\gamma}{dt} t + t^2 (\text{stuff}).$$

while

$$\det(\gamma) = \det \begin{pmatrix} \overset{1}{\overset{\|}{\gamma(0)_{00}}} + \left(\frac{d\gamma}{dt}\right)_{00} t + \cdots & & \overset{0}{\overset{\|}{\gamma(0)_{1n}}} + \left(\frac{d\gamma}{dt}\right)_{2n} t + \cdots \\ & \ddots & \\ & & \gamma(0)_{nn} + \left(\frac{d\gamma}{dt}\right)_{nn} t + \cdots \end{pmatrix}$$

$$= \det(\gamma(0)) + t(\qquad\qquad) + t^2(\text{stuff}).$$

$$= 1 + t \cdot \operatorname{trace}\left(\frac{d\gamma}{dt}\right) + t^2(\text{stuff}).$$

By assumption, γ is a curve in $SL_n(\mathbb{C})$, so $\det(\gamma(t))$ is a constant function $(-\varepsilon, \varepsilon) \to \mathbb{R}$. Hence all the terms in the Taylor expansion must be zero—in particular, the linear term

$$t \cdot \mathrm{trace}\left(\frac{d\gamma}{dt}\right)$$

equals zero, so we see that any tangent vector x to $e = I$ must satisfy

$$\mathrm{trace}(x) = 0.$$

(Here, we're identifying the collxn of vectors based at $I \in M_{n\times n}(\mathbb{C})$ with $M_{n\times n}(\mathbb{C})$ itself. This is always a good trick to use for a vector space like $M_{n\times n}(\mathbb{C})$.)

Since $\dim_{\mathbb{C}}(GL_n\mathbb{C}) = n^2 - 1$, and $\{x \mid \mathrm{trace}(x) = 0\}$ has $\dim_{\mathbb{C}} = n^2 - 1$, this is the whole tangent space:

<u>Prop</u> $T_e SL_n(\mathbb{C}) \cong \{x \in M_{n\times n}(\mathbb{C}) \mid \mathrm{trace}(x) = 0\}$.

Note that for any two matrices $x, y \in M_{n\times n}(\mathbb{C})$, we have

$$\mathrm{trace}([x, y]) = \mathrm{trace}(xy) - \mathrm{trace}(yx) = 0.$$

so in particular, $T_e SL_n(\mathbb{C})$ is closed under $[\ ,\ ]$.

Rmk If you don't like the implicit fxn thm, which tells you that

$$\dim_{\mathbb{C}} T_e SL_n(\mathbb{C}) = n^2-1,$$

a more high-powered proof: Given x s.t. $\mathrm{trace}(x) = 1$, consider

$$\exp(x) := I + x + \frac{x^2}{2} + \cdots$$

$$= \sum_{n\geq 0} \frac{1}{n!} x^n,$$

so

$$\det(\exp(tx)) = \det\left(\sum_{n\geq 0} t^n \frac{1}{n} x^n\right)$$

$$= \det(I) +$$

Ex

$$\mathcal{U}_n := \{A \in M_{n\times n}(\mathbb{C}) \mid A^*A = I\}$$

where $(A^*)_{ij} := \overline{(A_{ji})}$, so

$$(a)^* = (\bar{a})$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix}.$$

(That $A^*A = I$ means $\forall\ x, y \in \mathbb{C}^n$,

$$\langle Ax, Ay \rangle = (Ax)^* Ay = x^* A^* A y = x^* y = \langle x, y \rangle.)$$

What is $T_e\mathcal{U}_n =: \mathfrak{u}_n$?

Again writing $\gamma(t) = \gamma(0) + \frac{d\gamma}{dt} t + t^2(\text{---})$, $\gamma(0) = I = e$, have

$$\gamma(t)^*\gamma(t) = \Big(\overset{I}{\overset{\|}{\gamma(0)^*}} + \left(\tfrac{d\gamma}{dt}\right)^* t + \cdots\Big)\Big(\overset{I}{\overset{\|}{\gamma(0)}} + \left(\tfrac{d\gamma}{dt}\right) t + \cdots\Big)$$

$$= I + \left(I\frac{d\gamma}{dt} + \frac{d\gamma}{dt}^* I\right)t + t^2(\text{---})$$

Since $\gamma(t)^*\gamma(t) = I\ \forall t$, we have that $\frac{d\gamma}{dt}$ is a matrix x such that

$$x + x^* = 0. \qquad \text{ie,} \quad x = -x^*.$$

We have
$$[x,y]^* = (xy-yx)^*$$
$$= y^*x^* - x^*y^*$$
$$= (-y)(-x) - (-x)(-y)$$
$$= yx - xy$$
$$= -[x,y]$$

if $x, y \in u_n$, so $u_n \subset gl_n(\mathbb{C}) \subset gl_{2n}(\mathbb{R})$. is closed under $[\,,\,]$.

⚠ u_n is NOT a $\mathbb{C}$-vector space, since $(\,)^*$ is NOT a $\mathbb{C}$-linear operation.

We see $T_e U_n \subset \{x \mid x = -x^*\}$. One can prove this is infact an equality.

Ex $SU_n := \{A \in M_{n\times n}(\mathbb{C}) \mid A^*A = I, \det A = 1\}$

One can see $su_n := T_e SU_n \stackrel{\sim}{=} \{x \mid x = -x^*, \operatorname{trace}(x) = 0\}$, and can check

$$su_n \subset M_{n\times n}(\mathbb{C}) \cong M_{n\times n}(\mathbb{R}) \otimes \mathbb{C} \subset M_{2n\times 2n}(\mathbb{R})$$
$$A+iB \longmapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

is closed under brackets.

$\underline{\text{Ex}}$ $G = O_n = \{A \in M_{n\times n}(\mathbb{R}) \mid A^T A = I\}$.

Then $T_e G \subset \{x \mid x = -x^T\}$. (This is infact an equality, not just an inclusion.)

$$\begin{aligned} GL_n &\xrightarrow{\pi} \{\text{all symmetric matrices; ie, } B \in M_{n\times n}(\mathbb{R}) \text{ st } B^T = B\} \\ A &\longrightarrow A^T A. \end{aligned}$$

Note that by implicit fxn thm, if $D\pi_A$ is a syection @ each $A \in O_n$, we have that

$$\begin{aligned} \dim_{\mathbb{R}} T_A O_n &= \dim_{\mathbb{R}} T_A GL_n - \dim_{\mathbb{R}} \{B\} \\ &= n^2 - \left(\frac{n(n+1)}{2}\right) \\ &= \tfrac{1}{2}\left(2n^2 - n^2 - n\right) \\ &= \frac{n(n-1)}{2}. \end{aligned}$$

Consider

$$D\pi_I : T_e GL_n \longrightarrow \{B \text{ symmetric}\}.$$

What is the derivative?

$$\pi : A \longmapsto A^T A$$

$$D\pi|_I : x \longmapsto \underset{\overset{\parallel}{x + x^T}}{(I+x)^T(I+x) - I^T I} - \text{quadratic terms}$$

And obviously, since $\{B \text{ symmetric}\}$ is a vector space,

$$T_I\{B\} \cong \{B\}$$

and any B can be written $B = x + x^T$. So $D\pi_I$ surjection.

And then we have

$$YA \longmapsto A^T Y^T Y A = A^T A \quad , \text{ for any } Y \in O_n$$

$$\begin{array}{ccc} GL_n & \xrightarrow{\pi} & \{B\} \\ {\scriptstyle -\cdot Y} \uparrow & & \parallel \\ GL_n & \xrightarrow{\pi} & \{B\} \\ A & \longmapsto & A^T A \end{array}$$

hence $D\pi|_A = \underset{\overset{\uparrow}{\text{surj}}}{D\pi|_I} \circ \underset{\overset{\uparrow}{\text{isom}}}{D(- \cdot Y)|_A}$ when $Y = A^{-1}$.

Knowing $\dim_{\mathbb{R}} T_eO_n = \frac{n(n-1)}{2}$, we claim

$$T_eO_n \cong \{x \in M_{n\times n}(\mathbb{R}) \mid x = -x^T\}.$$

Pf: Fix

$$A : (-\varepsilon, \varepsilon) \longrightarrow O_n \hookrightarrow GL_n \subset \mathbb{R}^{n^2}$$
$$0 \longmapsto I.$$

Write $A(t)$ via Taylor expansion:

$$A(t) = I + \frac{dA}{dt}t + t^2(\text{stuff}).$$

$\forall t$, we have

$$I = A(t)^T A(t) = \left(I^T + \left(\frac{dA}{dt}\right)^T t + t^2(\text{stuff})^T\right)\left(I + \frac{dA}{dt}t + t^2(\text{stuff})\right)$$
$$= I^T I + \left(I^T\frac{dA}{dt} + \left(\frac{dA}{dt}\right)^T I\right)t + t^2\left(\substack{\text{other}\\ \text{stuff}}\right)$$

hence all coeffs of t must equal zero. We conclude

$$\frac{dA}{dt} + \left(\frac{dA}{dt}\right)^T = 0.$$

This gives $\mathfrak{o}_n := T_eO_n \subset \{x \text{ st } x = -x^T\} \leftarrow$ dimension $\frac{n(n-1)}{2}$.

Further, $[x,y]^T = y^Tx^T - x^Ty^T = yx - xy = -[x,y]$ if $x,y \in \mathfrak{o}_n$.