Quantum Subgroups and Higher Coxeter Graphs

by Adrian Ocneanu

Preliminaries: A model and sculpture

A model of a 6ft x 6ft x 6ft sculpture made at Penn State for the mathematics department. It illustrates (separately) both members of the McKay correspondence between **finite subgroups of SU(2)** and **simple Lie algebras**. It is the 24-cell, which I call the octacube, the 4th among the 4 dimensional regular solids. The rendering method, windowed radial stereographic projection, is new, and appears to interest the visual arts community as well as the popular press.


OCTACUBE
by Adrian Ocneanu


This sculpture is a gift from Jill
in memory of her husband Kermit
who was killed in the terrorist
in New York City on September 11,

On the **subgroups of SU(2)** side its nodes are the binary tetrahedral subgroup of SU(2), and using the mid rooms as well, the binary octahedral subgroup, which correspond to the affine graphs E_6 and E_7. The edges and surfaces have natural subgroup interpretations, with the holes given by a play of lights in regular solids as drawn by Leonardo da Vinci 500 years ago.

On the **Lie algebras** side its nodes are the root system of type D_4, and using part or all the mid rooms as well, the root systems of type $B_4 = C_4$ and F_4. The sculpture also illustrates the Weyl groups of these types, as well as the reduction projection from D_4 to $B_3 = C_3$ and G_2.

The 24 spheres surrounding a sphere in the lattice packing can be seen on the sculpture as well.

In a short paper in 1990, McKay made the following crucial observation. The Cartan matrix C of a unimodular affine Lie algebra has the form $C = 2 - \Delta_\Gamma$ where Γ is an ADE graph and Δ_Γ is its adjacency matrix. Any such graph Γ is obtained from a subgroup $G \subset SU(2)$ as the fusion graph $\Gamma = \Gamma_G$ (analog of Cayley graph) for tensoring the irreducible representations $\mathrm{Irr}\, G$ with $\sigma|G$, the 2 dimensional irreducible σ of $SU(2)$ restricted to G. As $\dim(\sigma|G) = \dim\sigma = 2$, we get by Perron-Frobenius $||\Delta_\Gamma|| = 2$ with a unique eigenvalue 2, and thus the Cartan matrix $C = 2 - \Delta_\Gamma$ is positive with one degenerate eigenvector.

The fact that there are graphs Γ with $||\Delta_\Gamma|| = 2$ which do not appear above, the tadpoles, was not addressed but will be discussed in our talk.

Thus there is a correspondence between
$G \subset SU(2)$ subgroup with $\dim(\sigma|G) = 2$
$\leftrightarrow$
$C \geq 0$ **degenerate** Cartan matrix for an affine simple Lie algebra.

We shall describe **quantum subgroups** G which we introduced, for which there is a correspondence between
$G \subset SU(2)_N$ with $\dim(\sigma|G) = [2]_N = 2\cos(\pi/N)$
$\leftrightarrow$
$C > 0$ **nondegenerate** Cartan matrix for a simple Lie algebra.

Thus the ADE (nonaffine) graphs have naturally irreducible objects as vertices and have edges given by tensoring. The quantum subgroups of $SU(2)$ are already quite different from the (classical) subgroups of $SU(2)$, with D_{odd} and E_7 different from the other ADE's. When we go to $SU(3), SU(4)\dots$ the quantum subgroup classification will be very different, and simpler than, the classification of the corresponding (classical) subgroups.

Part I: Extending a monoidal tensor category

The data for a monoidal tensor category consists of:

- A set of (irreducible) objects $\{X, Y, Z, \ldots\}$

- Euclidean vector spaces $\mathrm{Hom}[X \otimes Y, Z]$ (the fusion) with a trivial object 1,

- Coefficients (6j symbols) for changing base between

$$\begin{aligned}
&\mathrm{Hom}[(X \otimes Y) \otimes Z, T] = \\
&= \bigoplus_U \mathrm{Hom}[X \otimes Y, U] \otimes \mathrm{Hom}[U \otimes Z, T] \\
&\mathrm{Hom}[X \otimes (Y \otimes Z), T] = \\
&= \bigoplus_V \mathrm{Hom}[X \otimes V, T] \otimes \mathrm{Hom}[Y \otimes Z, V]
\end{aligned}$$

(the 6 j's are the 6 objects X,Y,Z,T,U,V involved)

The main axiom is a pentagonal identity which expresses the naturality of base change. From this one obtains symmetry relations: Each object X has a conjugate $\overline{X}$ with $X \otimes \overline{X} \ni 1$. The Hom spaces $\mathrm{Hom}[X \otimes Y, Z]$ have the symmetry group S_3 acting on the triangle with edges X, Y, Z (Frobenius reciprocity), e.g.

$$\mathrm{Hom}[X \otimes Y, Z] \approx \mathrm{Hom}[X, Z \otimes \overline{Y}] = \overline{\mathrm{Hom}[Z \otimes \overline{Y}, X]}$$

The axioms are modeled after the irreducible representations of a finite or compact group, less the commutativity.

An important additional data is a braiding, in which there is a distinguished isomorphism

$$\varepsilon : \mathrm{Hom}[X \otimes Y, Y \otimes X]$$

for each pair of objects X, Y, which commutes with the fusion.

Modeled after the bimodules coming from subfactors, it is interesting to extend such a tensor category in the same way in which, in topology, one goes from the **group of loops** at a base point to the **groupoid of paths** on a manifold.

We give a set of labels, the **types** $\{A, B, C, \dots\}$ and the objects have each a **type**, which is a pair of labels (source and range) ${}_A X_B$. The axioms remain as before, except for the fact that **intertwiners are defined only for matching types**, as in $\mathrm{Hom}[{}_A X_B \otimes {}_B Y_C, {}_A Z_C]$.

For a fixed type A the objects $\{{}_A X_A\}$ form a monoidal tensor category. The **extension** problem for a monoidal tensor category starts by labeling the objects X of the category as $\{{}_A X_A\}$. The problem is then **finding all the possible other compatible types** $B, C, \dots$. The natural conditions are the following

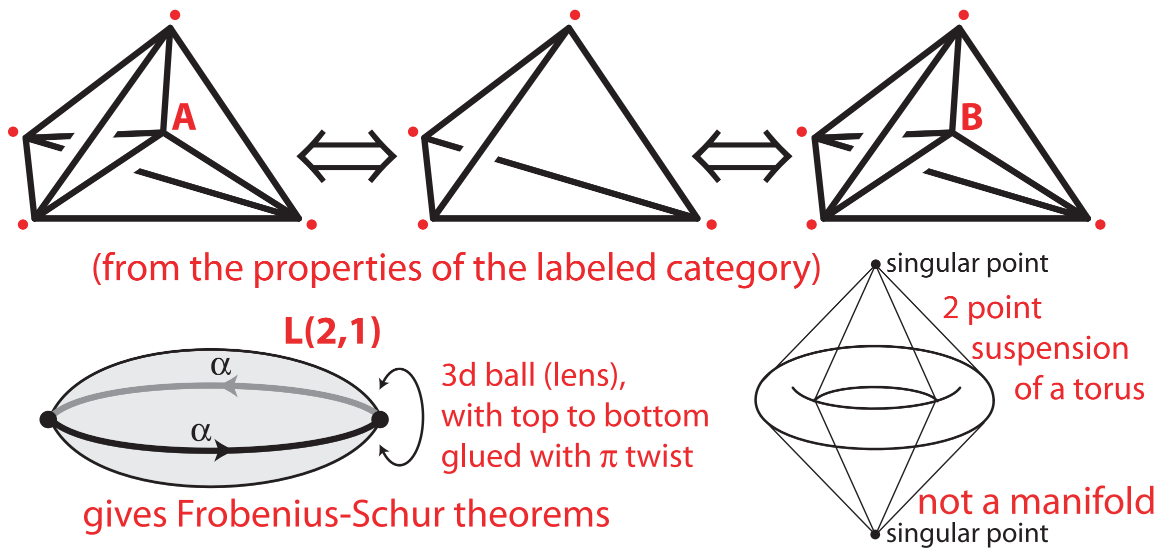
• The **nondegeneracy** condition. Any $({}_AX_B) \otimes_B \left({}_B\overline{X}_A\right)$ decomposes into the given $A-A$ objects.

• The **nonredundancy** condition. For any distinct labels B, C there is no invertible ${}_BX_C$ (otherwise C is a relabeling of B).

Then $\mathrm{dist}(B,C) = \min_X \log[{}_BX_C]$ is a **distance** between vertices.

We called this maximal extension the **maximal atlas** of the given $A-A$ system. The idea behind the name is that the **objects of different types**, e.g. in the case of a group the **group elements** and the **group irreducibles**, are providing **alternative descriptions**, or **maps**, of the **same structure**.

The problem is defined in such simple terms (given a tensor category of $A-A$ objects, find all the possible types $B, C, \ldots$ and objects of type $A-B$, $A-C$, $B-C$, etc.) that it seems either trivial or impossible. In fact it lies in the interesting domain in between, it can be solved in several general contexts, and leads to new objects and results there.



Thus switching labels **each 3-manifold gives a theorem in representation theory**, stating that a certain quantity is the computed with group elements is the same when computed with group representations. The sphere S^3 gives

$$|G| = \sum_{\sigma \in \mathrm{Irr}\, G} |\sigma|^2$$

while the projective plane or lens space $L(2,1)$ gives the Frobenius-Schur theorem

$$|\{g \in G : g^2 = 1\}| = \sum_{\sigma \in \mathrm{Irr}\, G : \sigma \otimes \sigma \ni 1} \pm |\sigma|$$

Part II: Quantum Subgroups

The problem of the maximal extension of a tensor category can be solved in the case of the elements or the irreducible representations of a finite group G, and led to the subgroups H of G twisted by a 2-cocycle.

The natural next step is the maximal extension of the tensor category coming from a quantum group at a root of 1. The objects which we obtained this way are called by analogy **quantum subgroups**.

From the time of Euler on, numbers, then functions and afterwards whole mathematical structures appear to have natural q-deformations.

The number $n = 0, 1, 2, \ldots$ deforms to the quantum number

$$[n] = (q^{n/2} - q^{-n/2})/(q^{1/2} - q^{-1/2})$$

so e.g. $[3] = q^{-1} + 1 + q$. From a vector space V we define (formally) q^V with elements $\{q^v, v \in V\}$ satisfying $q^v q^w = q^{v+w}$. Just like quantum numbers we have now quantum vectors, $[v] = (q^{v/2} - q^{-v/2})/(q^{1/2} - q^{-1/2})$.

The main step in quantizing $SU(2)$, and similarly simple Lie groups, is to replace the diagonal vector space H by q^H and the relation $[e, f] = h$ by $[e, f] = [h]$.

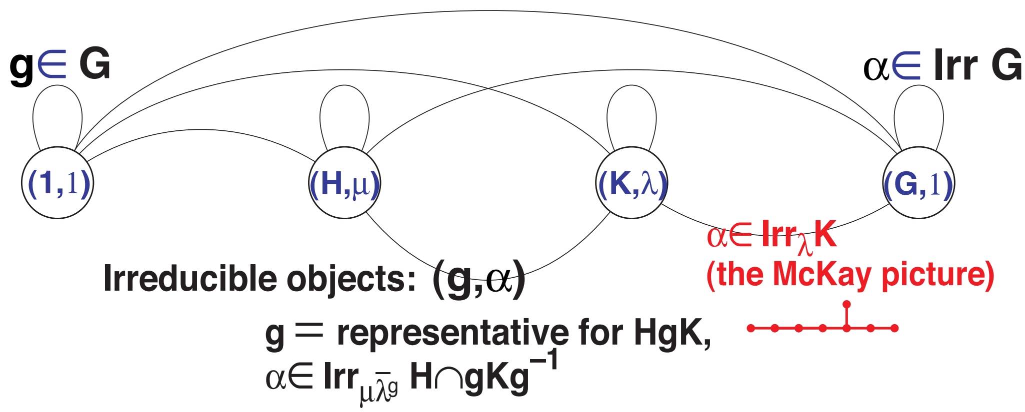
The dimension of the irreducible $\sigma_n \in \mathrm{Irr}\, SU(2)$ of degree n becomes $[n + 1]$. At a root of 1, when $q = e^{2\pi i/N}$, we have $[N] = 0$ (N is called the Coxeter number), and by a quotienting procedure the quantum group $SU(2)_N$ remains with only a **finite** number of irreducible representations $\sigma_0, \sigma_1, \ldots, \sigma_{N-1}$. These form a **braided tensor category**, so in view of our previous discussion, the natural problem is to find the maximal extension from $\mathrm{Irr}\, SU(2)_N$ viewed as $A - A$ objects to all the possible $B, C, \ldots$ labels and corresponding objects.

The main result is that the types of the quantum subgroups of $SU(2)_q$ with $q^N = 1$ are precisely those ADE graphs which have Coxeter number N . Thus, e.g. when N is odd, the only label is $A = A_{N-1}$ while for $N = 30$ the labels are $A = A_{29}, D_{16}$ and E_8. Thus $SU(2)_{odd}$ has no nontrivial quantum subgroups while $SU(2)_{30}$ has 2 nontrivial quantum subgroups, the quantum analogs of the binary dihedral and the binary icosahedral subgroups of $SU(2)$ studied by Felix Klein.

The maximal atlas for a finite or compact group G

Vertices = (H,μ) = subgroup H of G + scalar 2 cocycle μ on H

(μ Schur multiplier->projective representations of H)

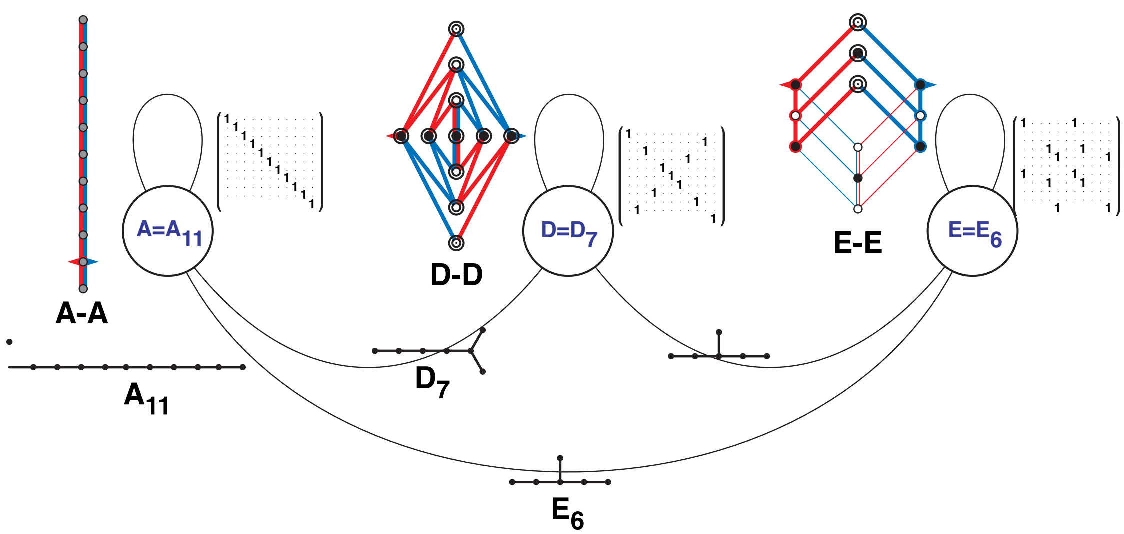


The maximal atlas of a quantum group at a root of 1 (Coxeter N) (A.O.): a quantum analog of the McKay picture

$G = SU(2)_{10}$ withCoxeter number 12

Vertices = ADE graphs with Coxeter number N

By analogy we call these the quantum subgroups of G



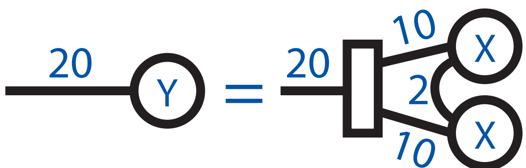
Note that the quantum subgroups appear in this construction as a set of irreducible objects (representations) with Hom spaces for tensoring. Their ”internal structure” remains an open problem.

The Kleinian **invariant theory** has a very interesting quantum analog. The **degrees** of the quantum invariants correspond to the entries in the **modular invariant matrix** defined first in physics.

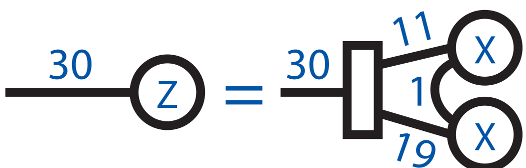
Invariants for classical and quantum subgroups of SU(2)

is the spin k/2 irreducible of SU(2)

The E_8 (binary icosahedral) subgroup of SU(2)

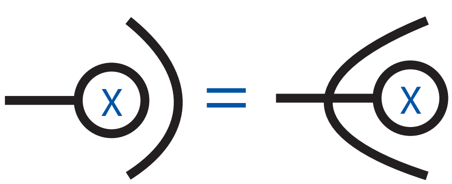


(Y = Hess(X))

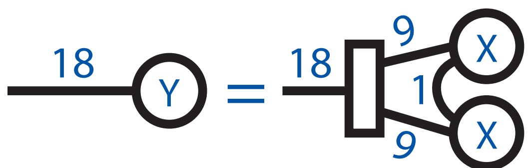


(Z = Jac(X,Y))

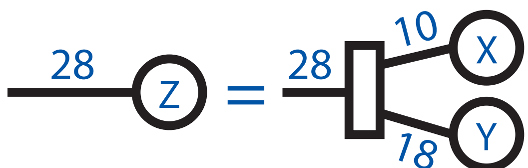
polynomials in X,Y,Z modulo
$X^5+Y^3+Z^2=0$



The E_8 subgroup of quantum $SU(2)_{28}$ ($q^{30}=1$)



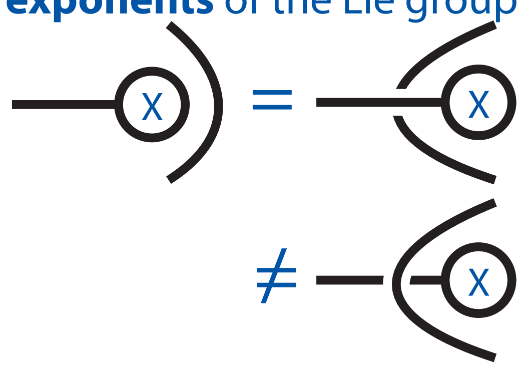
(Y = Jac(X,X))



(Z = X Y)

no invariants other than 1,X,Y,Z
(other polynomials are 0)

(0,10,18,28)+1=(1,11,19,29) are the **exponents** of the Lie group E_8



We have (A.O., 1993)

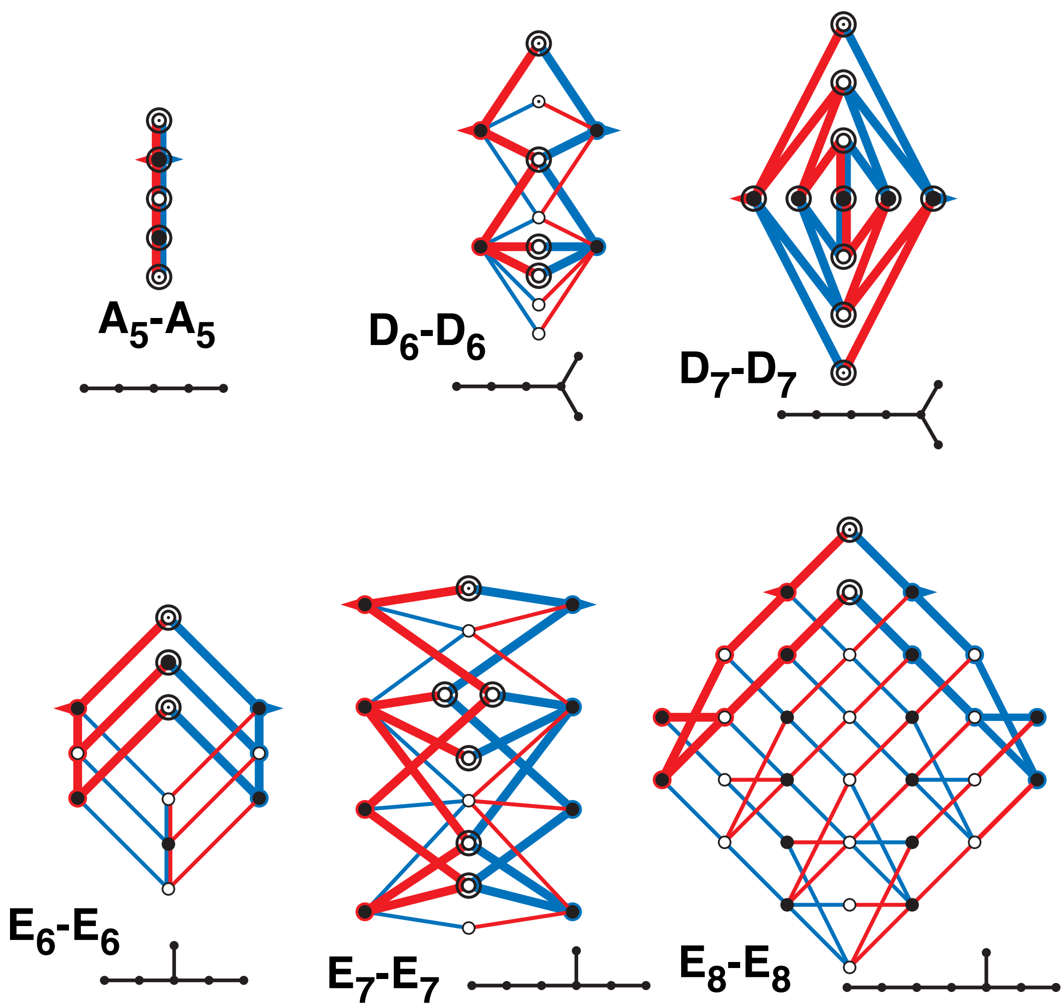
• the ADE graphs have finitely many quantum symmetries. Their number is

graph	A_n	D_n $n{=}2n'$	D_n $n{=}2n'{+}1$	E_6	E_7	E_8
cox. no.	n	$2n-2$	$2n-2$	12	18	30
q.symm.	n	$2n-2$	$2n-1$	12	17	32

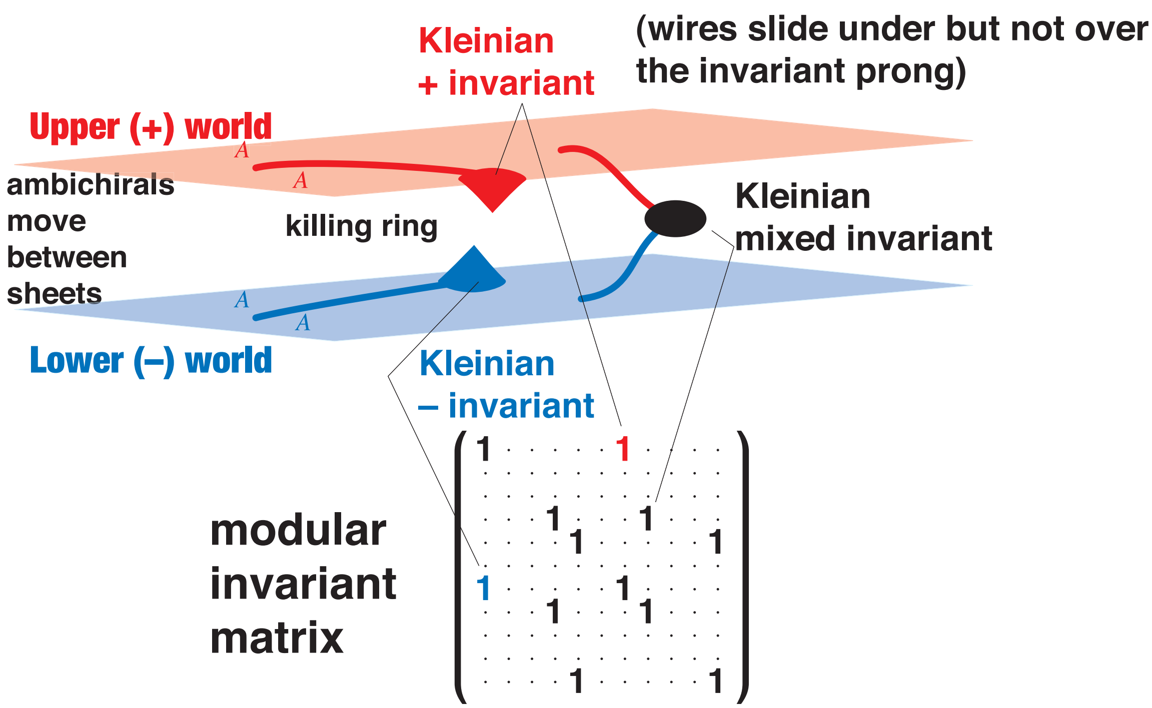
• the **affine** ADE graphs have **finitely parametrizable** quantum symmetries.

• the quantum symmetries of graphs of norm > 2 are **wild**.

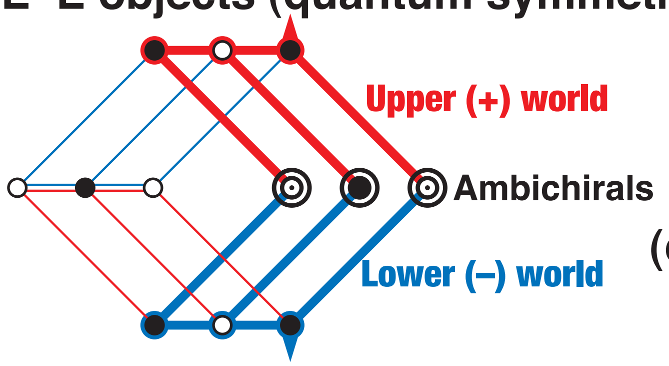
Each ADE graph has two nontrivial **generating quantum symmetries** with coefficients complex conjugate to each other. The quantum morphisms of ADE graphs and between them are the following (A.O.)



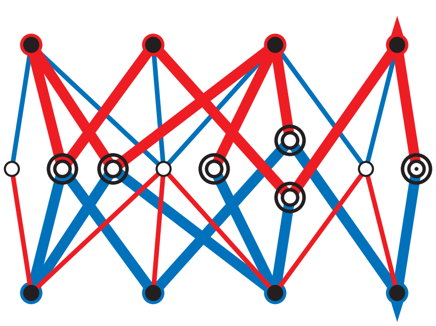
The chiral worlds picture



E–E objects (quantum symmetries)



E_6-E_6
no ambichiral twist
(quantum subgroup =
type I in physics)



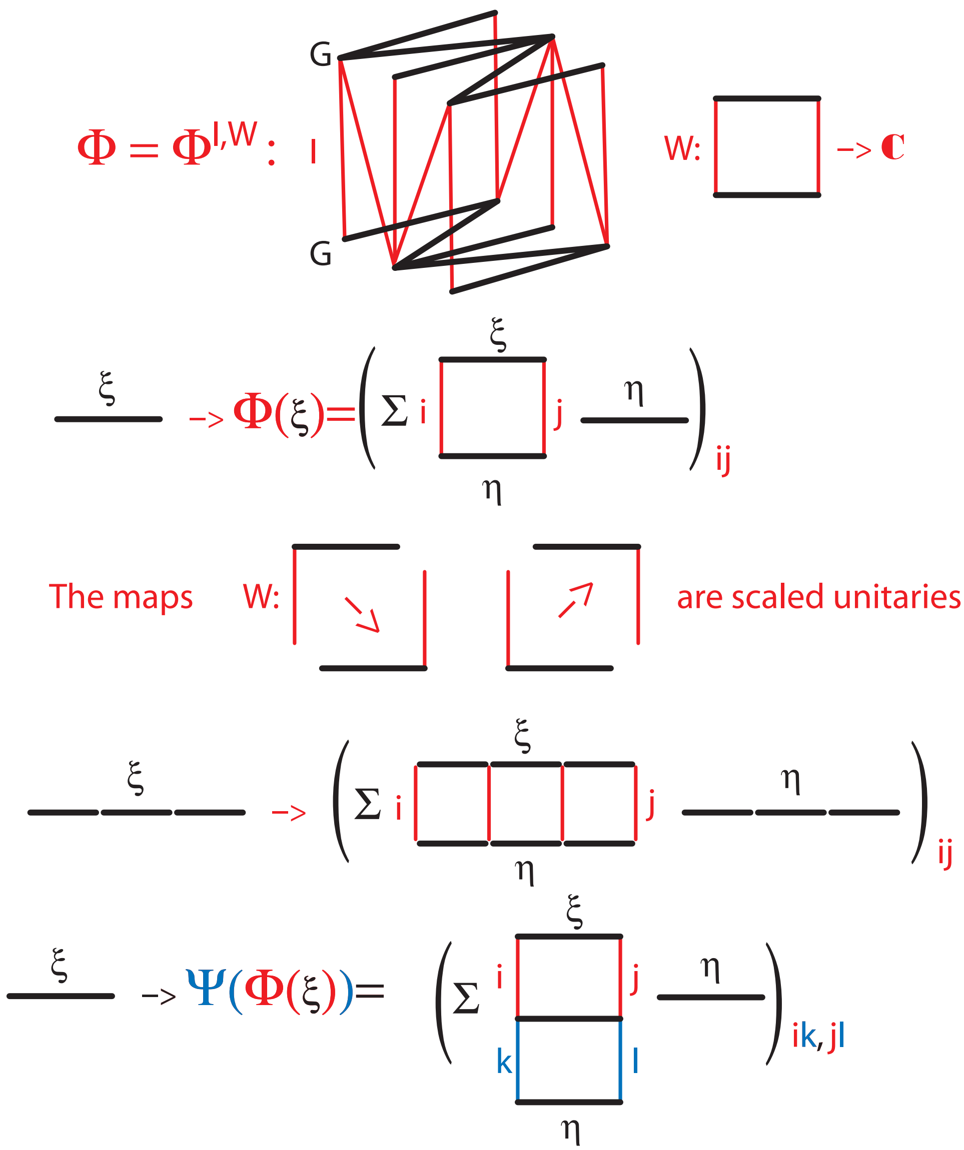
E_7-E_7
D_{10} with ambichiral twist
(module only =
type II in physics)

We shall present now an elementary approach, the **quantum symmetries of graphs**, which starts from scratch. Let G be a graph, here typically ADE or affine ADE. Such graphs have few symmetries in the usual, or classical, sense.

In quantum mechanics a particle is no longer punctual, but spread around; the points in this room X are replaced by linear combinations of points $\mathbb{C}^X$ (too big!), or rather $L^2(X, \mathbb{C})$.

In the same spirit, let us replace the Edge G by $\mathbb{C}^{\mathrm{Edge}\, G}$ and look for its automorphisms.

Quantum symmetries of graphs



Change basis in vertical graphs and decompose into irreducibles

The quantum symmetries of a graph of type A_n are isomorphic, as a tensor category, with the irreducible representations of the quantum group $SU(2)_q$ at the $n-1$ root of 1 This gives the most elementary realization of quantum group cutoffs.

For a finite group G the maximal atlas is labeled by pairs $(H,[\mu])$ of a subgroup $H \subset G$ up to inner conjugacy and a scalar 2-cocycle (Schur multiplier) μ on H.

Let G_l denote a semisimple group G of cut off by the WZW construction at a root of 1 with level (= highest degree of irreducibles) l. In view of the previous discussion it is natural to call the labels of the **maximal atlas** coming from G_l the **quantum subgroups** of G_l.

Remark that the TQFT language in which we introduced the **quantum subgroups** is very close to the tensor category into which they were later rewritten by Kirillov and Ostrik (the edges labels are the objects of the category, the Hilbert spaces of triangles are the morphisms, etc.) They obtained a new very interesting characterization of the distinction between subgroups and modules (type I and type II.)

The main problem as in all TQFT is not the language used but is **(i) the construction of rich examples**, **(ii) the understanding of the inner structure of the objects**, leading to **(iii) classification results**.

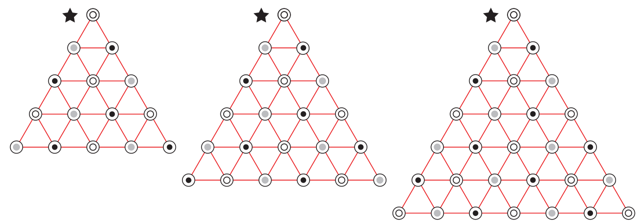
We showed that the quantum subgroups give raise to, and can be alternatively described by, quantum groupoid structures called double triangle algebras. These have been studied by Robert Coquereaux with collaborators, who found very interesting properties and new aspects of them. In fact, quantum groups (more generally, quantum groupoids) describe precisely the topological properties of rhombuses. Adding an involution corresponds to allowing to flip rhombuses on the other side.

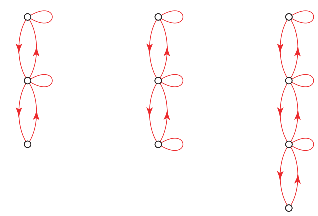
The **quantum subgroups** at roots of unity of **quantum groups** have provided an unexpectedly rich structure. We have developed this structure for quantum subgroups of $SU(2)$ with methods which work without essential modifications for any **nondegenerately braided tensor category**.

Answering the champagne bottle problem of Zuber, which asked which are the **higher $SU(3)$ analogs of the $SU(2)$ Coxeter ADE graphs**, we reformulated the problem as the problem of classifying the **quantum subgroups of $SU(3)$** (the reformulation was kindly accepted by Zuber).

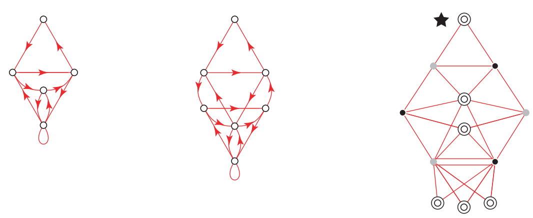
We could classify the **quantum subgroups of $SU(3)$** using the classification of modular invariants of $SU(3)$ by Terry Gannon, and the list of candidates remarkably guessed almost perfectly with "computer aided flair" by diFrancesco and Zuber, except for a graph which was not a quantum subgroup. and a later addition of orbifolds by
diFrancesco, Petkova, Pierce and Zuber)

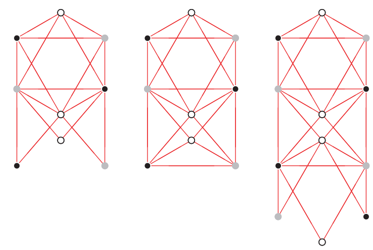
$\mathbf{SU(3)_k}$ orbifold series





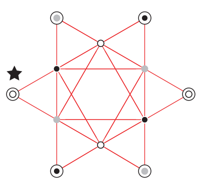
$\mathbf{A_4}$ $\mathbf{A_5}$ $\mathbf{A_6}$ ... $\mathcal{A}_4^c$ $\mathcal{A}_5^c$ $\mathcal{A}_6^c$ ...

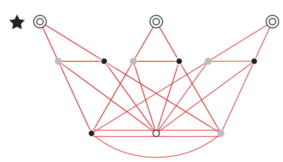


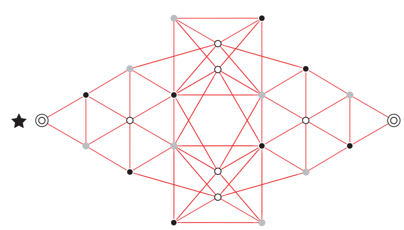


$\mathcal{A}_4/3$ $\mathcal{A}_5/3$ $\mathbf{A_6/3}$ ... $3\mathcal{A}_4^c$ $3\mathcal{A}_5^c$ $3\mathcal{A}_6^c$ ...

$\mathbf{SU(3)_k}$ exceptional subgroups

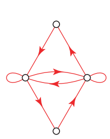


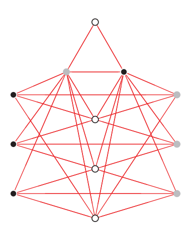


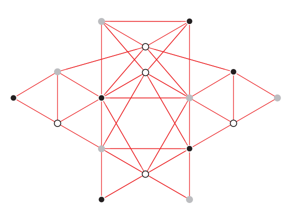


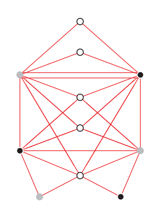
$\mathbf{E_5}$ $\mathbf{E_9}$ $\mathbf{E_{21}}$

$\mathbf{SU(3)_k}$ exceptional modules









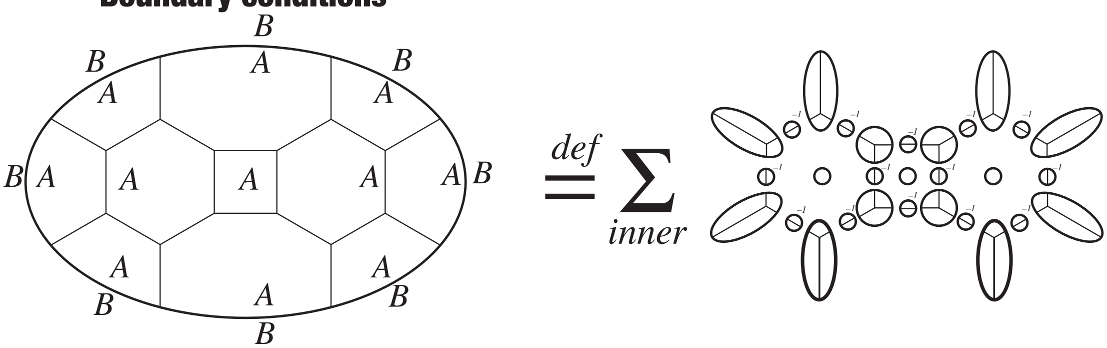
$\mathcal{E}_5/3=(\mathcal{E}_5)^c$ $\mathcal{E}_9/3=(\mathcal{E}_9)^c$ $(\mathcal{A}_9/3)^t$ $(\mathcal{A}_9/3)^{tc}$

We developed a simple new method (**cells** related to 6j symbols) to characterize quantum subgroups.

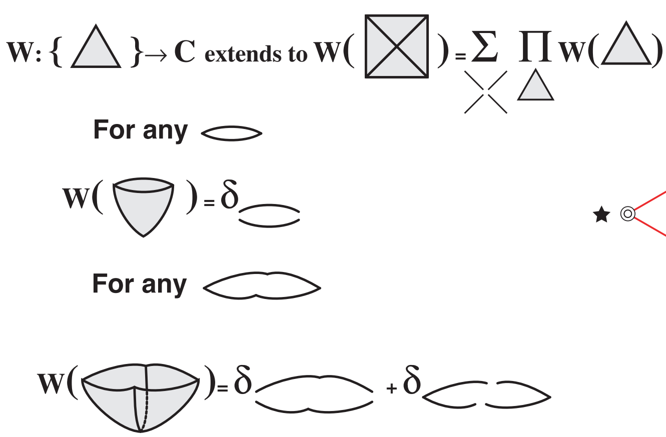
Modules, braiding and modular theory

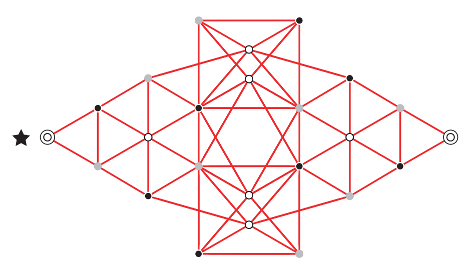
Given A-A objects construct the A-B objects (modules, boundary conditions)

Boundary conditions



Cells for a graph of a subgroup or module of $SU(3)_N$ (A.O.)





$\mathbf{E_{21}}$

(normalization coefficients omitted)

On the constructive side we developed a method, **modular splitting** for constructing graphs from modular invariants (an equivalent observation was made independently by F.Xu). We developed a simple **bootstrap method** for differentiating between subgroups and modules (type I and type II). We characterized the **modules associated to a given subgroup** by the **ambichiral twist**. Using these results and methods **the classification of quantum subgroups of $SU(3)$ is only mildly computational and does not require any machine help**.

For the classification of the **quantum subgroups of $SU(4)$**, while Gannon's algorithms are very efficient for determining modular invariants up to fairly high levels, the modular invariant classification is not known. The nonlinearity of the modular splitting yields **upper bounds for the Coxeter number of exceptional subgroups**.

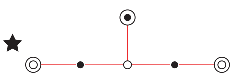
This gives the following rigidity result, with a computable (if very large) upper bound. **For any semisimple Lie group, there are no exceptional subgroups beyond a certain (computable) level (A.O.)**. Thus there are finitely many orbifold series and finitely many exceptionals.

The **actual highest level of exceptional quantum subroups is unexpectedly small**: for $SU(2)$ it is 28, for $SU(3)$ it is 21 and for $SU(4)$ it is the surprisingly low 8. We constructed with intensive computation the exceptional subgroup of $SU(4)_8$ and its module, which are the first examples of quantum subgroups which do not come from any known CFT construction such as conformal inclusions. There are as well **unexpectedly few exceptional subgroups**: 2 for $SU(2)$, 3 for $SU(3)$ and 3 for $SU(4)$. Compare this to the huge number of exceptional subgroups of the classical $SU(4)$.

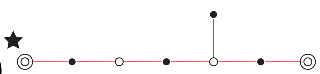
Exceptional subgroups

$\mathbf{SU(2)_k}$

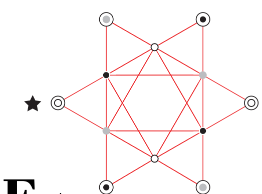
$\mathbf{E_6\,(E_{10})^\star}$



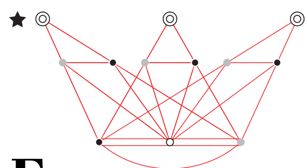
$\mathbf{E_8(E_{28})^\star}$



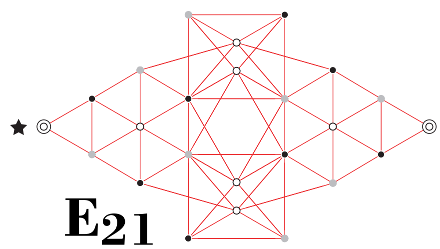
$\mathbf{SU(3)_k}$



$\mathbf{E_5}$

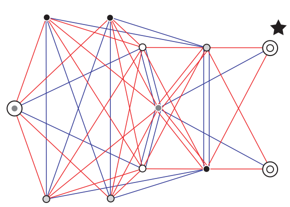


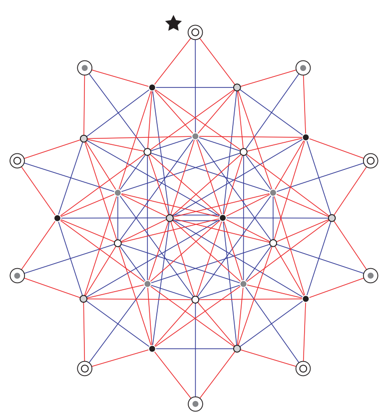
$\mathbf{E_9}$

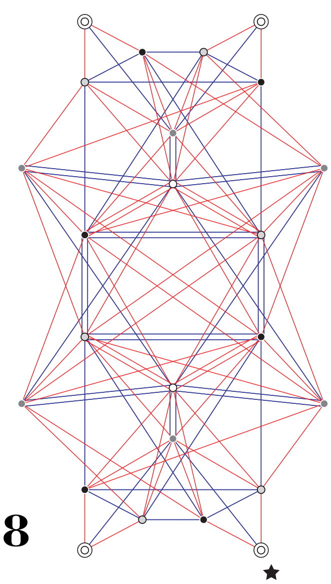


$\mathbf{E_{21}}$

$\mathbf{SU(4)_k}$







$\mathbf{E_4}$ $\mathbf{E_6}$ $\mathbf{E_8}$

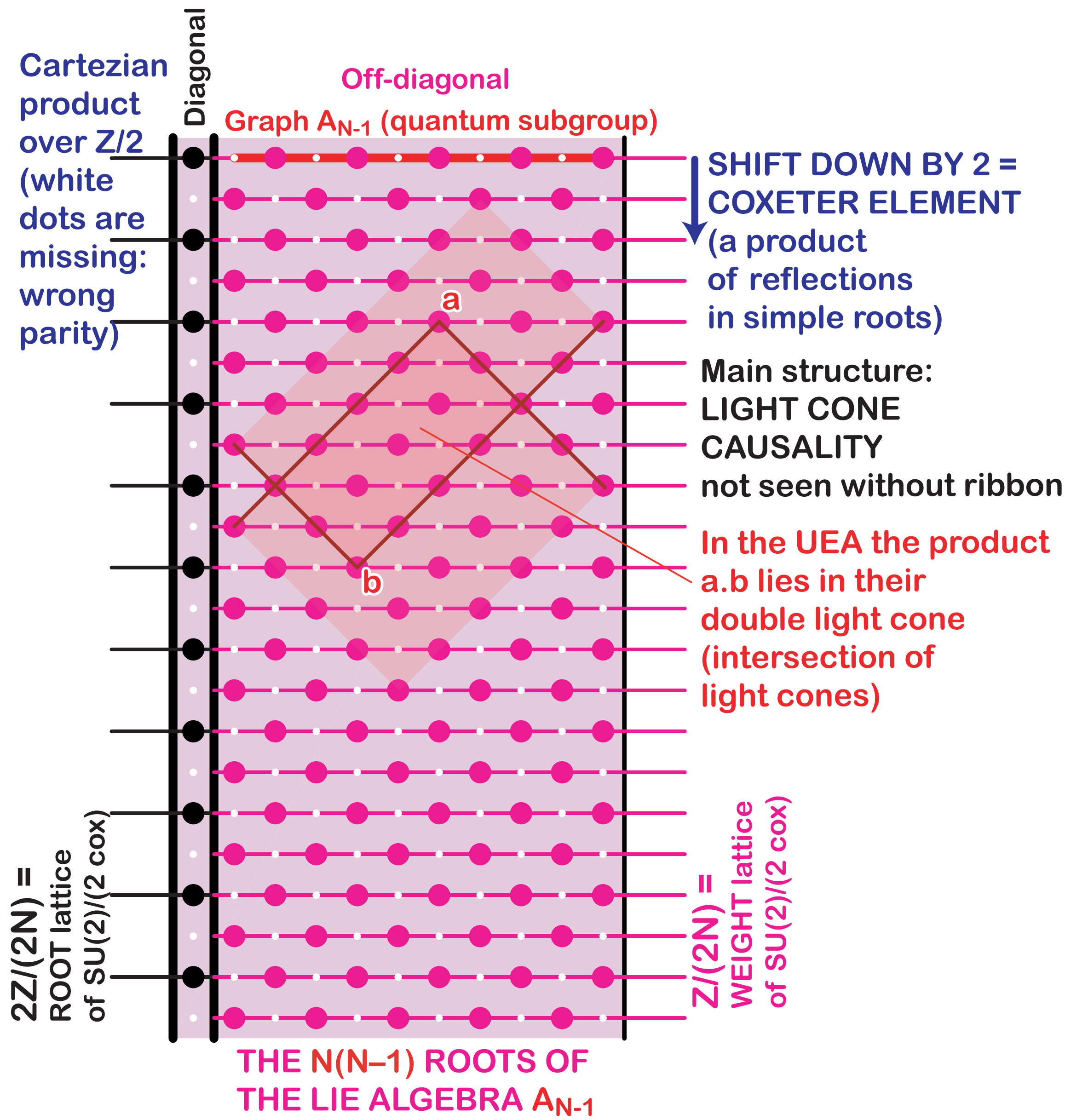
Thus **the quantum subgroup picture**
- classifies the solutions of the **boundary CFT** problem
- classifies **boundary TQFTs** for a given TQFTs
- answers problems set by Zuber et al. about **higher Coxeter graphs**
- classifies the solutions of the **boundary statistical mechanical problem** of Zuber, Pierce, Petkova et al.
- admits an elementary description as **quantum symmetries of graphs**
- admits an easy test using **cells on graphs**
- answers several apparently unrelated **main problems in operator algebras**
- explains the **off-diagonal entries of the modular invariants** in several different ways
- provides a **machinery for the effective construction of the quantum subgroups** of a nondegenerately braided object, such as quantum groups at roots of 1.

Part III: Simple Lie Groups from Quantum Subgroups of $SU(2)_N$.

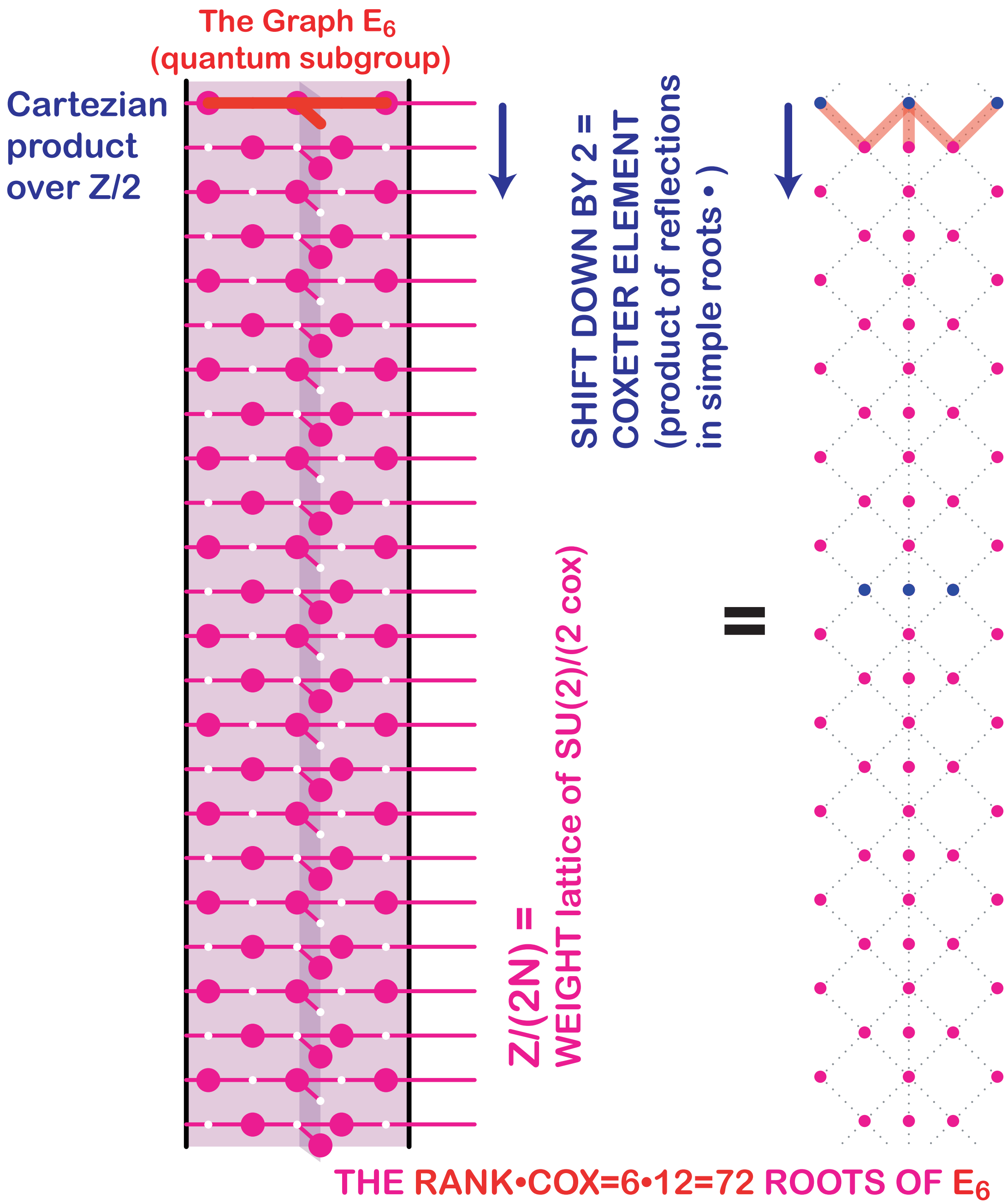
We want to **rewrite and simplify the classical construction and representation theory of simple Lie groups** using a new approach based on **quantum subgroups of** $SU(2)$.

The **main problem** with the traditional approach is the **splitting** of Lie algebras into **the upper and the lower triangular parts**, which makes the construction of the universal enveloping algebras unnecessarily complicated. Rather than impose our will, we should let the constructions ask for the appropriate structure.

So we start with a new way to handle the n x n matrices. (movie)

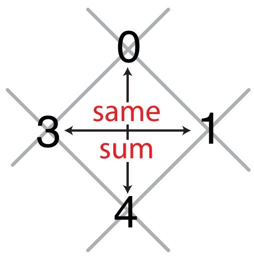


The roots of any ADE graph are built the same way.

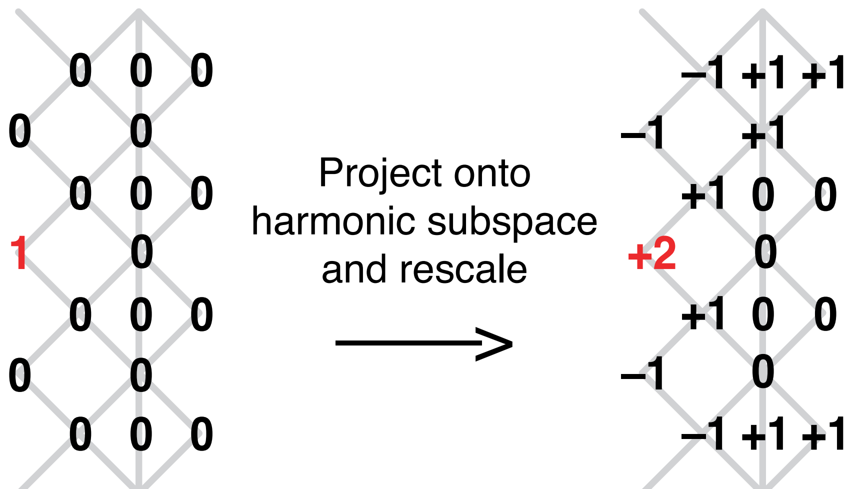


The linear space $\mathbb{C}^{\mathrm{Roots}}$ is too big. As the space of roots is a cartezian product of two graphs, we have a graph Laplacian (= adjacency matrix) Δ_{hor} for the horizontal graph and Δ_{vert} for the vertical graph. We call **harmonic** the functions f : Roots $\to \mathbb{C}$ with $\Delta_{hor}(f) = \Delta_{vert}(f)$.

The **root space** consists of the **harmonic functions** on the ribbon. The roots are obtained by projecting the base of $\mathbb{C}^{\text{Roots}}$ on the harmonic functions.



The root space consists harmonic functions on the cartezian product graph $\Delta_{hor}(f) - \Delta_{vert}(f) = 0$

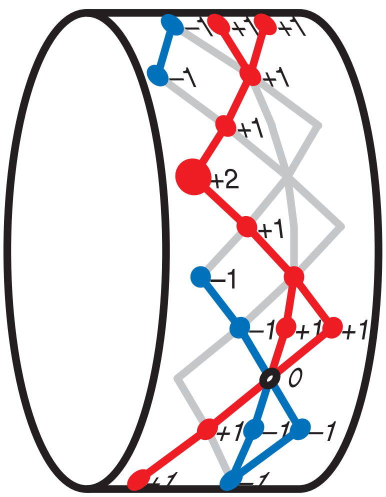


Dirac function at a root

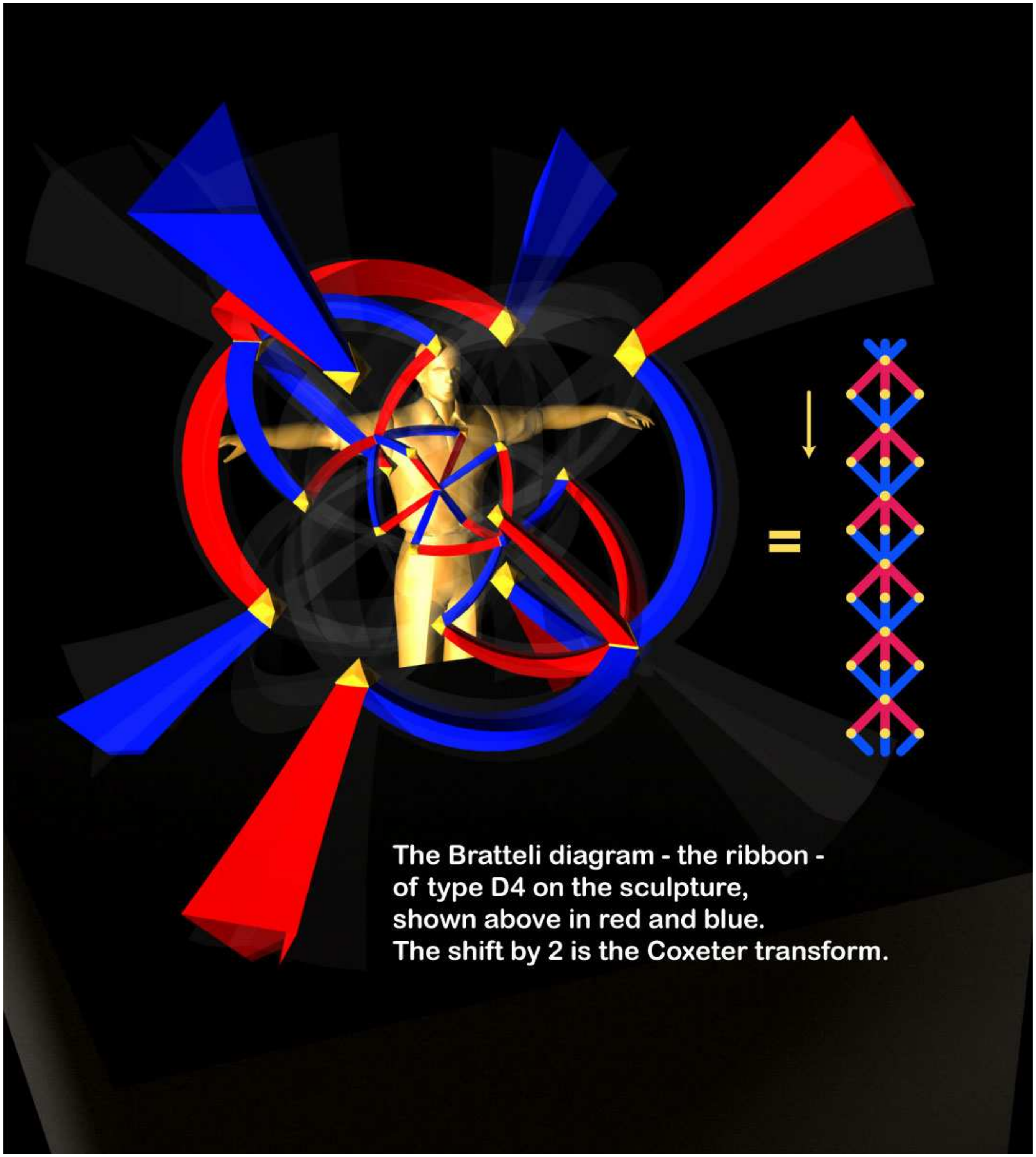
inner product of a root with the other roots

- **Weights** are given by integer valued **harmonic functions** on the ribbon. Roots $\to \mathbb{Z}$.

All the roots and weights have been constructed at the same time, without a choice of a simple root base.



root shell



The Bratteli diagram - the ribbon -
of type D4 on the sculpture,
shown above in red and blue.
The shift by 2 is the Coxeter transform.

In fact we shall prove that **any integral coefficient basis of the universal enveloping algebra makes by its mere existence a different choice**: it distinguishes a **Coxeter element**.

This will show that the **ribbon structure** which we introduce on semisimple Lie algebras is in fact the **most natural and canonical** structure possible.

The main structure on the ribbon is the **light cone causality** i.e. the product in the UEA between two terms lives in the **double cone causality region** between them.

Canonical Bases for Universal Enveloping Algebras

According to the PBW theorem, the universal enveloping algebra (UEA) of a simple Lie group, e.g. $SU(n)$ has off diagonal base

$$: \prod_{ij} e_{ij}^{n_{ij}} := \prod_{ij} e_{ij}^{n_{ij}} + \text{ lower degree terms}$$

Equivalently one makes some some **choice of order** in each product.

One wants a **canonical choice** of UEA base to which **all automorphisms of the Lie algebra extend**. This is impossible though from the following argument, given here for convenience in the case of $SU(n)$.

In the UEA we have $e_{12}.e_{23} - e_{23}.e_{12} = [e_{12}, e_{23}] = e_{13}$. With coefficients mod 2, **either** $e_{12}.e_{23} = e_{13} + \ldots$ and we write $e_{12} < e_{13} < e_{23}$ **or** $e_{23}.e_{12} = e_{13} + \ldots$ and we write $e_{23} < e_{13} < e_{12}$. This **breaks the symmetry** , since a **Weyl group element** which interchanges e_{12} with e_{23} **cannot extend** to the UEA basis . These orderings for all noncommuting pairs **arrange the roots on a ribbon** and **distinguish the Coxeter element of the ribbon**. So the **ribbon approach** is **the most canonical construction possible.**

This is why the structure of **multiplicity and intertwiner spaces of representations** is **much simpler and more natural** on the **ribbon**.

A base construction more canonical than the usual lexicographic ordering was first done by **Lusztig** starting from a **choice of simple base** and makes the UEA base independent of the ordering of the simple base . One starts with an order dependent construction due to **Ringel** using **quiver representations** for the **upper diagonal part**. **Lusztig** then compensates for the ordering of quivers by using a **braiding**. The end result depends on the choice of simple base.

What is the structure behind the ribbon?

• The idea is that the quantum subgroup S of $SU(2)_N$ contains the information for putting together copies of $SU(2)$ into a simple Lie group G, with its quantum deformation, universal enveloping algebra with a canonical base, representations, etc..
• From $SU(2)_N$ itself we construct $SU(N)_q$ and e.g. from the **binary icosahedral** subgroup of $SU(2)_{30}$ we construct the **Lie group of type** $(E_8)_q$.
• While we may work elementarily with an ADE graph Γ it is useful to view it as the graph of irreducible representations $\Gamma = \mathrm{Irr}\, S$ of a quantum subgroup or module (we call it subgroup) S of $SU(2)_N$. This means that we have a well defined tensor product

$$\mathrm{Irr}\, S \times \mathrm{Irr}\, SU(2)_N \to \mathrm{Irr}\, S$$

defined in a natural way.
• A simple Lie algebra, e.g. $su(N)$, is obtained

by putting together copies of $su(2)$ with combinatorics given by a root system $\{r_{ij}\}$. There is a diagonal part, in which r_{ij} corresponds to $h_{ij} = e_{ii} - e_{jj}$. **We construct the root geometry from the fusion (i.e. tensoring multiplicities) for quantum subgroups.**

• The graph ADE is the McKay (or Cayley) graph for tensoring the subgroup irreducibles $\mathrm{Irr}\, S$ with the generator σ_1 of $SU(2)_N$. Paths of length n between $\alpha \in \mathrm{Irr}\, S$ and $\beta \in \mathrm{Irr}\, S$ are a base of

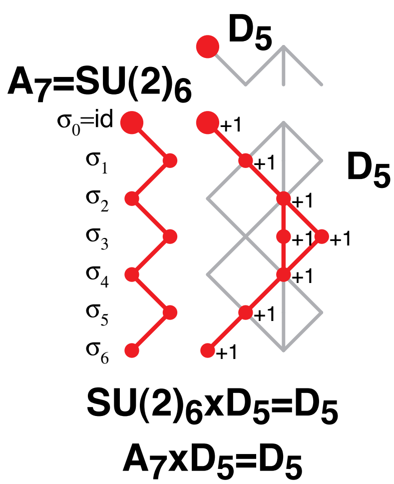
$$\mathrm{Hom}_S[\alpha \otimes \sigma_1^{\otimes n}, \beta].$$

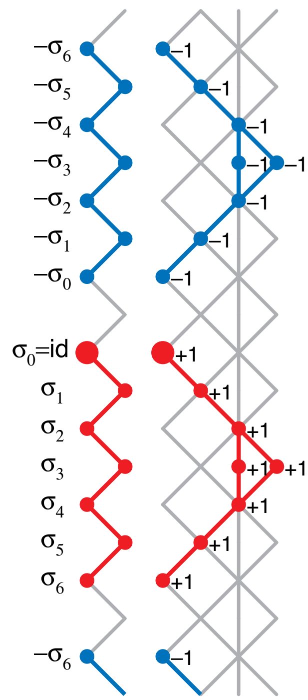
Inside this there is the linear subspace which we call **essential paths** corresponding to the highest weight $\sigma_n \subset \sigma_1^{\otimes n}$

$$\mathrm{Hom}_S[\alpha \otimes \sigma_n, \beta],$$

with dimension called **fusion number** $N^{\beta}_{\alpha,n}$.

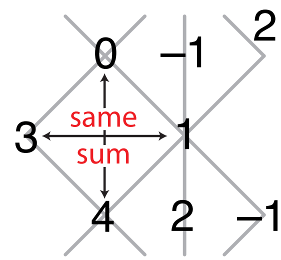
fusion





due to associativity, fusion functions on the cartezian product graph are harmonic:

$\Delta_{hor}(f) - \Delta_{vert}(f) = 0$



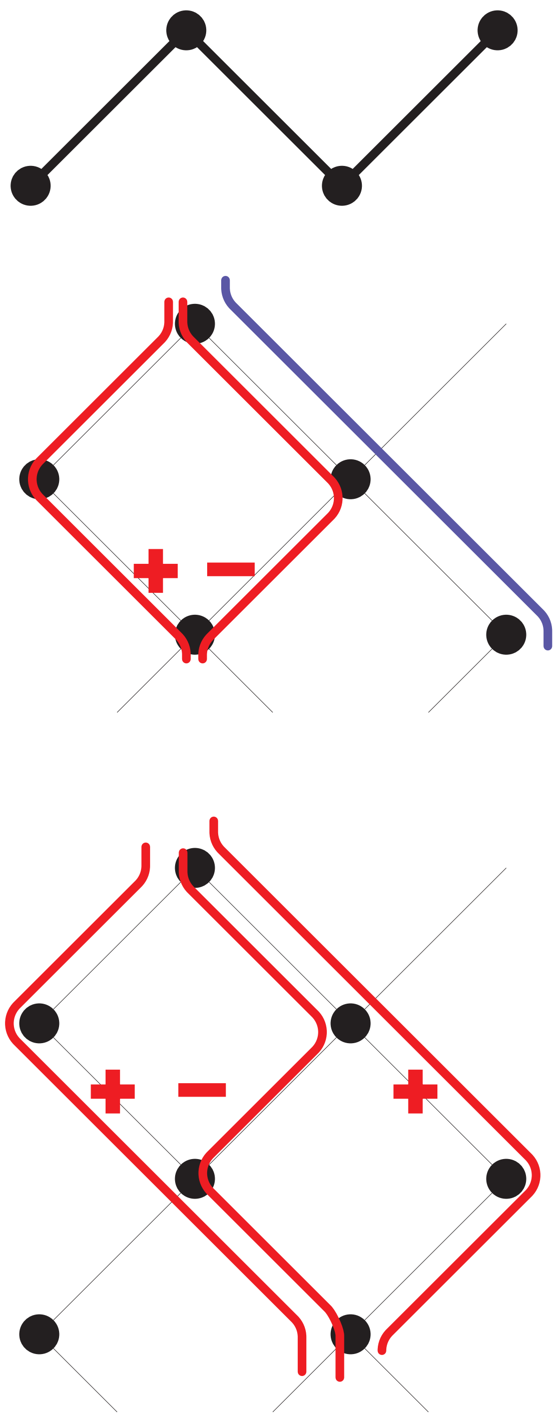
Elementarily, the k-th **contraction** on paths on a graph acts (up to a normalization) by

$$\mathrm{contr}_k : \xi = (\xi_1, \ldots, \xi_n) \mapsto$$

$$\delta_{\xi_k, \xi_{k+1}^{-1}} (\xi_1, \ldots, \widehat{\xi_k}, \widehat{\xi_{k+1}}, \ldots, \xi_n)$$

An **essential path** is a linear combination of paths for which all the contractions are 0. This condition reflects the fact that the irreducible $\sigma_n \subset \sigma_1^{\otimes n}$ is not contained in lower degree (i.e. shorter) $\sigma_1^{\otimes m}$ for $m < n$.

essential paths



• When we concatenate essential paths

$$\xi \in \mathsf{Hom}_S[\alpha \otimes \sigma_n, \beta]$$

and

$$\eta \in \mathsf{Hom}_S[\beta \otimes \sigma_m, \gamma]$$

we obtain

$$\xi \circ \eta \in \mathsf{Hom}_S[\alpha \otimes \sigma_n \otimes \sigma_m, \gamma].$$

Then we project it onto essential paths, corresponding to $\sigma_{n+m} \subset \sigma_n \otimes \sigma_m$, to get

$$\xi \cdot \eta \in \mathsf{Hom}_S[\alpha \otimes \sigma_{n+m}, \gamma].$$

This **product of essential paths** is the foundation of the whole construction of Lie groups from quantum subgroups.

• We construct a linear category with objects

$$\mathsf{Roots} = \mathbb{Z}/(2N) \times_{\mathbb{Z}/2} \mathsf{Irr}\, S$$

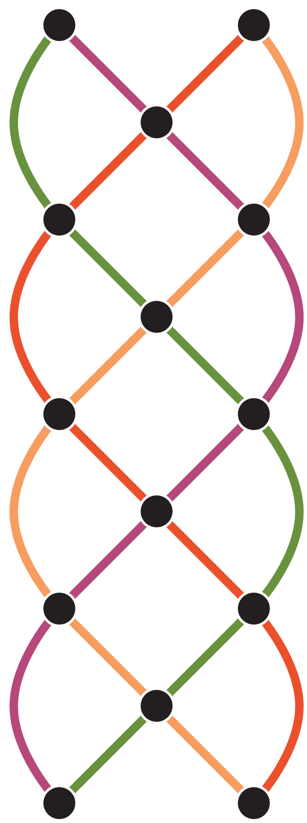
(taken with multiplicities) and homomorphisms given by essential paths on the product graphs

$$\mathsf{Hom}[(k, \alpha), (l, \beta)] := \mathsf{Hom}_S[\alpha \otimes \sigma_{l-k}, \beta].$$

Recall that we have defined a product of essential paths, corresponding to concatenation followed by highest weight projection corresponding to $\sigma_{n+m} \subset \sigma_n \otimes \sigma_m$.

• The kernel of a homomorphism has again a kernel (these are not vector space maps) and remarkably **after 6 terms any exact sequence closes** ($2N$ steps higher, but our vertical coordinate has period $2N$.) This gives the **hexagons in the root lattice**. The **snake lemma** in homology theory becomes the **root system of $SU(4)$**.

Snake lemma (homological algebra)



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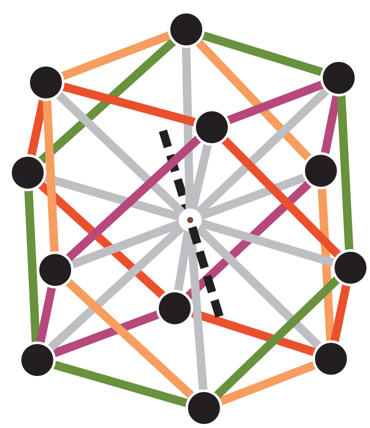
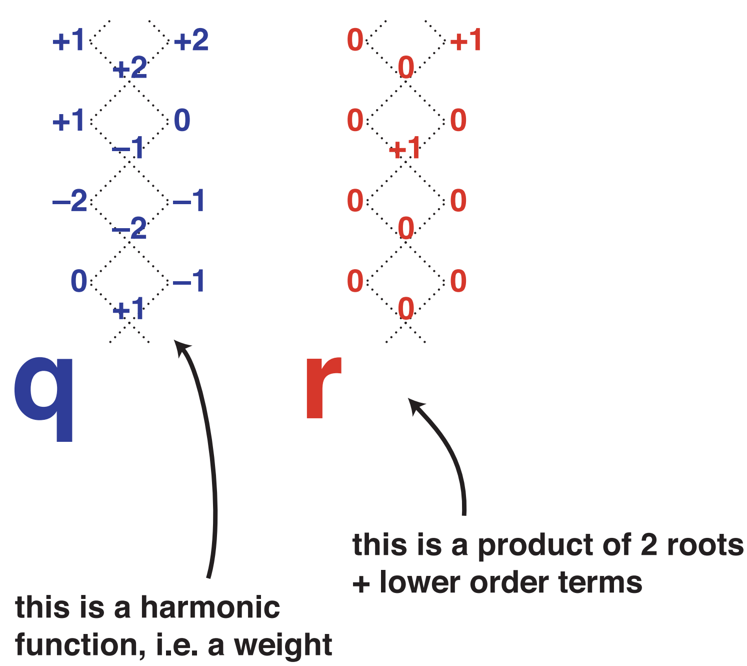
Root system of type A_3 (SU(4))



Thus **homology theory** has a **crystallographic component**.

• The off-diagonal **canonical base** (we do not single out the upper triangular part) is labeled by multiplicities $\mathbf{n}\colon \text{Roots} \to \mathbb{N}$. We denote the corresponding base element by a formal power $\mathbf{r}^{\mathbf{n}}$, which is a **natural choice** $\mathbf{r}^{\mathbf{n}} =: \prod_i r_i^{n_i} :$, **intrinsic in the ribbon construction**, for the product $\prod_i r_i^{n_i}$ modulo lower order commutants.

a canonical base element



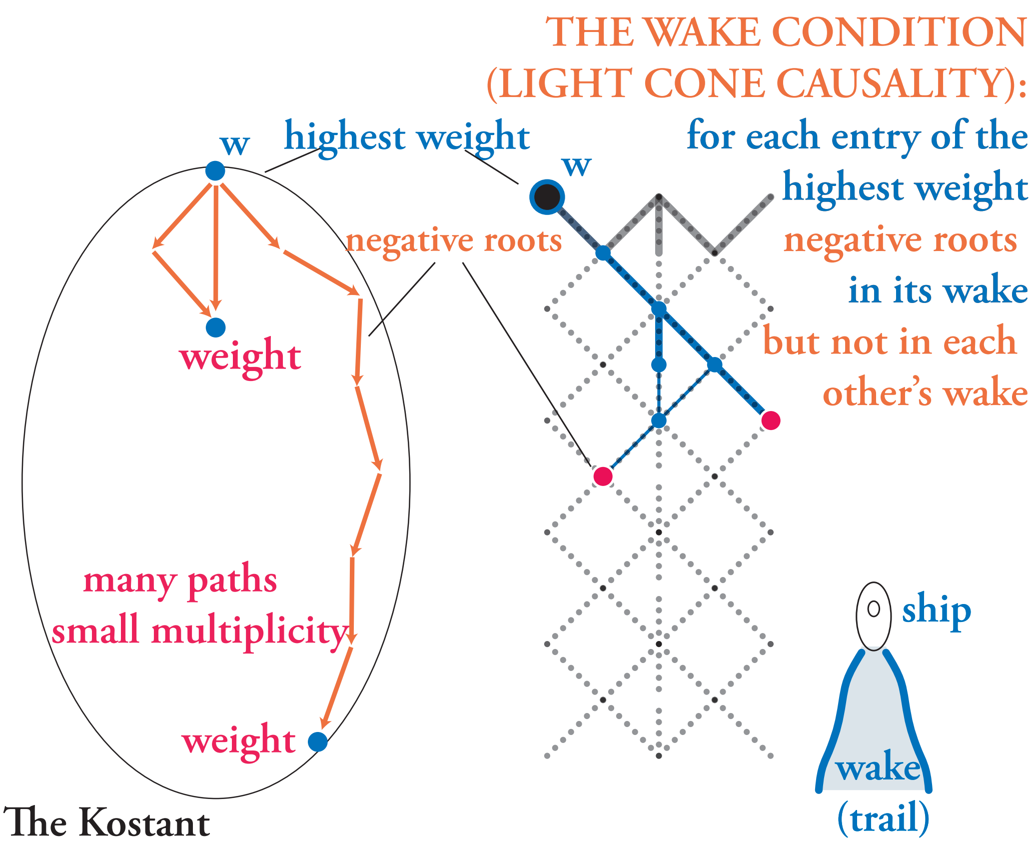
• It remains to define the product $\mathbf{r^n r^m}$. We use the Hom's defined before, make and count extensions adapting Ringel's beautiful idea of the Hall algebra with coefficients counting linear maps over a field with q elements.

The number of extensions is counted over the field with q elements, and is a polynomial in q.

Then the **number** q **of elements in the field** becomes the **deformation parameter** q in the Lie algebra.

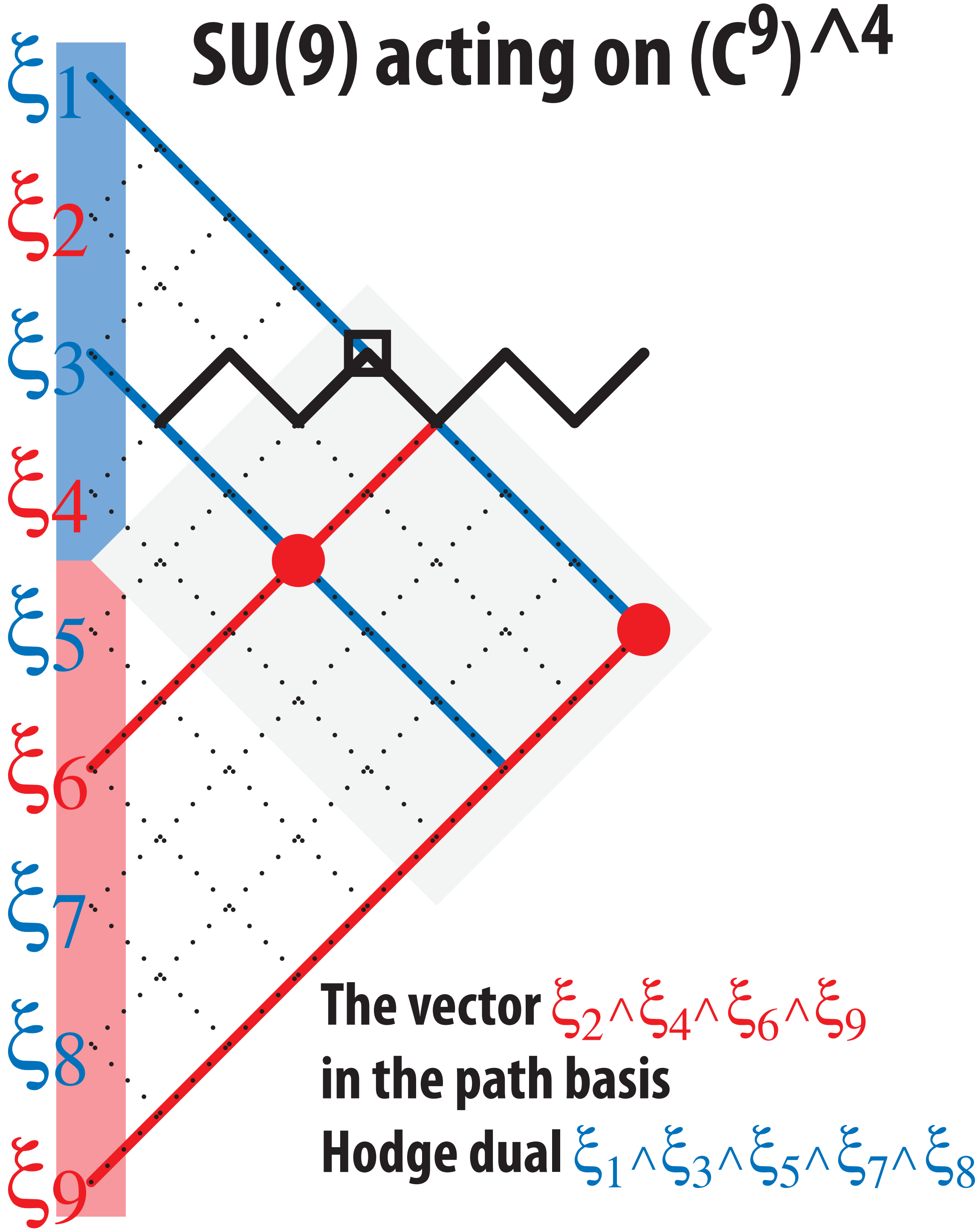
• The ribbon construction provides a new **path basis** for the **representations** of the simple Lie groups.

Multiplicities of an irreducible representation of a simple group G

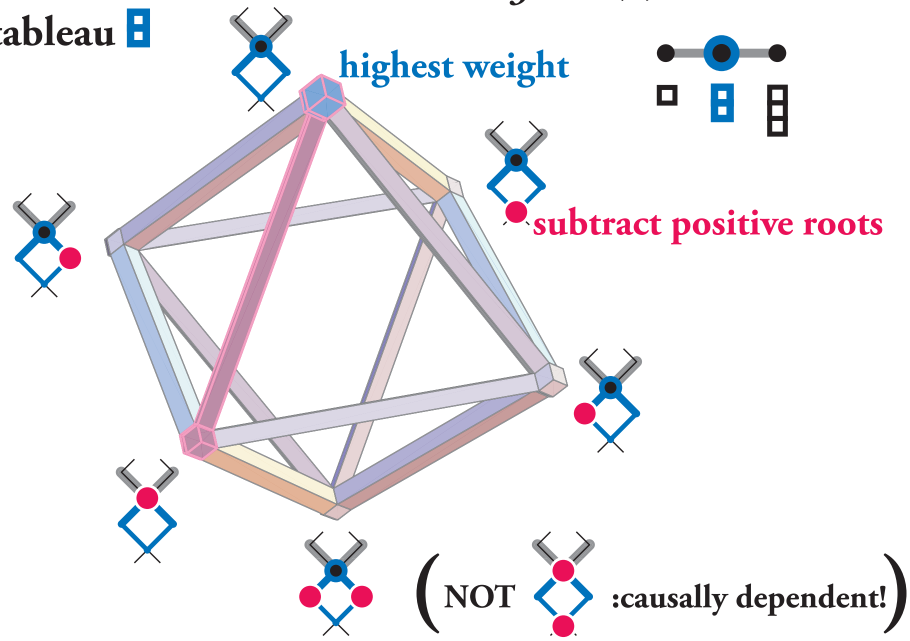


The Kostant multiplicity formula: count all possible ways to add negative roots. Correct this by adding with alternating signs paths from transforms of highest weight by the (huge) Weyl group.

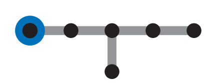
The wake condition on the band chooses the correct multiplicity: no corrections are needed

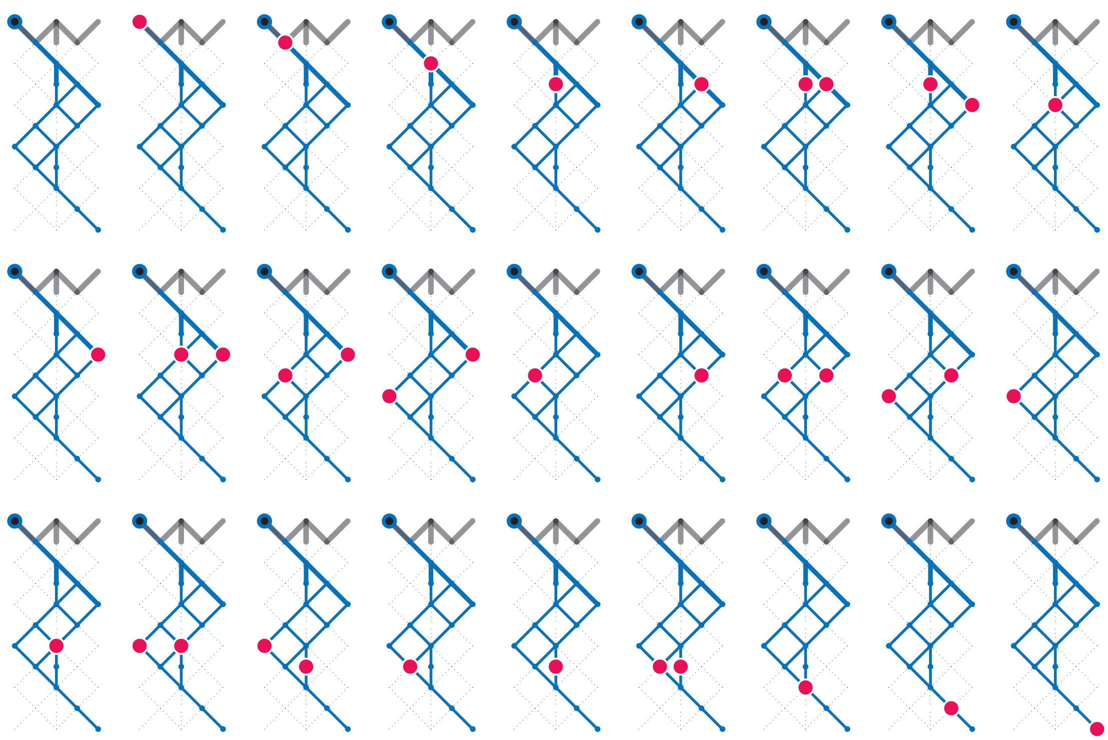

SU(9) acting on $(C^9)^{\wedge 4}$
ξ_1
ξ_2
ξ_3
ξ_4
ξ_5
ξ_6
ξ_7
ξ_8
ξ_9
The vector $\xi_2 \wedge \xi_4 \wedge \xi_6 \wedge \xi_9$
in the path basis
Hodge dual $\xi_1 \wedge \xi_3 \wedge \xi_5 \wedge \xi_7 \wedge \xi_8$

The canonical basis of the irreducible of A_3 = su(4) with Young tableau

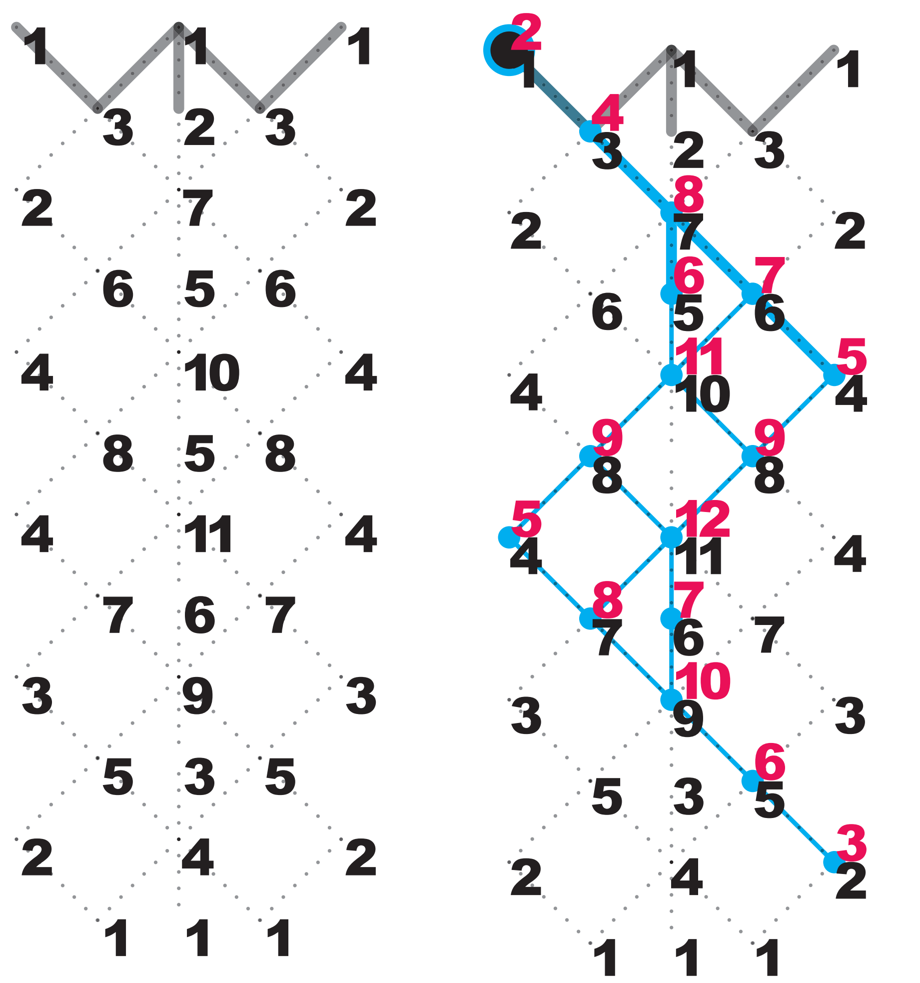


The canonical basis of the standard irreducible of E6 (27 dim)





The fundamental formula of Weyl has a simple interpretation in terms of essential paths.



The Weyl vector+representation 9 12

The Weyl vector 4

[9][12] / [4]
=[1]+[9]+[17]
= 27

Part IV:
Higher Analogs of Simple Lie Groups from Quantum Subgroups of $SU(K)_N$.

The construction and representation theory of simple Lie groups from quantum subgroups of $SU(2)$ was simple and natural enough to extend to quantum subgroups of $SU(K)_N,\ K > 2$.

• Each subgroup and module give raise to a Euclidean system of **generalized roots** and **generalized weights**. These lattices are **new even in the simplest cases**.

- **The $SU(3)_1$ Lattice Theta Function**

The theta function of the lattice corresponding to $SU(3)_1$ is

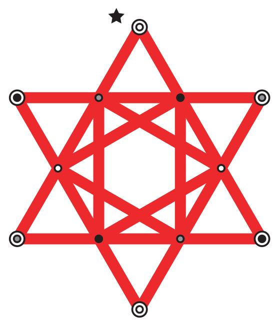
$$\theta(q) = \sum_{m=0}^{\infty} N(m) q^m$$
$$= 1 + 32q^3 + 60q^4 + 192q^7 + 252q^8 + \dots$$

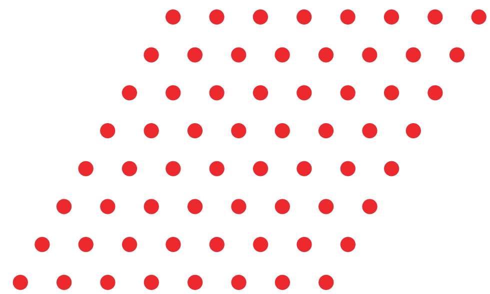
where $N(0) = 1$ and for $m > 0, m \equiv 0$ or 3 mod 4 with $m = 2^{n_2} 3^{n_3} 5^{n_5} \dots$

$$N(m) = 4 \cdot$$
$$\cdot (2^{2n_2} - (-1)^{n_3(3-1)/2 + n_5(5-1)/2 + \dots}) \cdot$$
$$\cdot \frac{3^{2(n_3+1)} - (-1)^{(n_3+1)(3-1)/2}}{3^2 - (-1)^{(3-1)/2}} \cdot$$
$$\cdot \frac{5^{2(n_5+1)} - (-1)^{(n_5+1)(5-1)/2}}{5^2 - (-1)^{(5-1)/2}} \cdot \dots$$

If $m \equiv 1$ or $2 \mod 4$ let $N(m) = 0$.

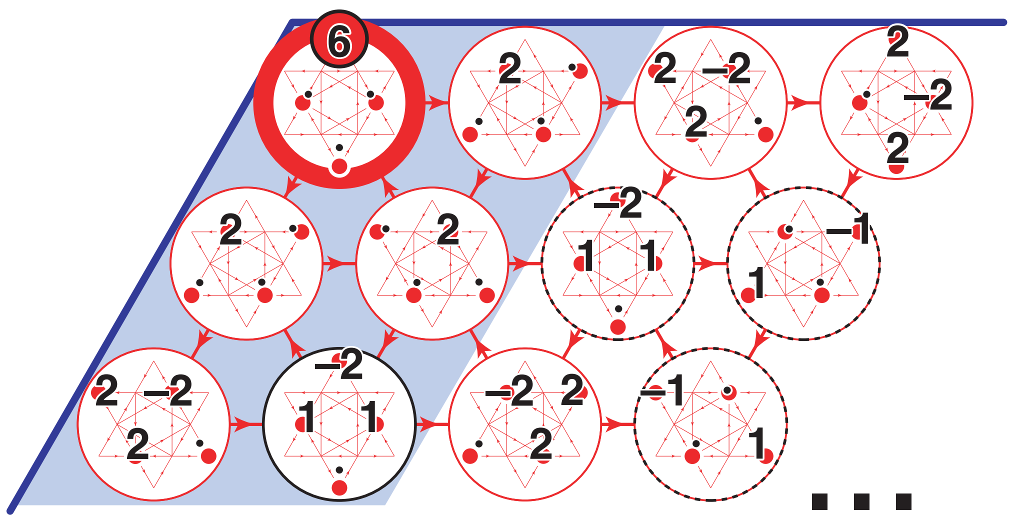
This is a very interesting **multiplicative modular function**.



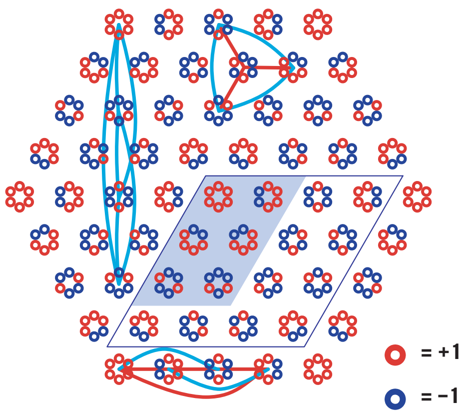


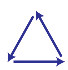
The E_5 q.subgroup of $SU(3)_5$ (cox 8) $\times_{Z/3}$ Weights $SU(3)/8$

= 256 higher roots in 24 dim Euclidean space

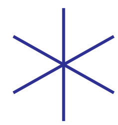


Scalar product of a higher root with the others

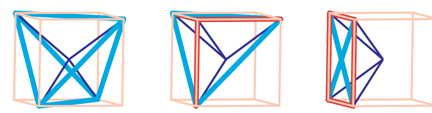


From $SU(3)_5$ =  **we obtain**
16 generalized roots in R^6:
the lattice D_6^+, never before used
in representation theory
(note: D_8^+ is the lattice E_8).
All other generalized lattices
are new.

Instead of the usual
Lie algebra lattice hexagons



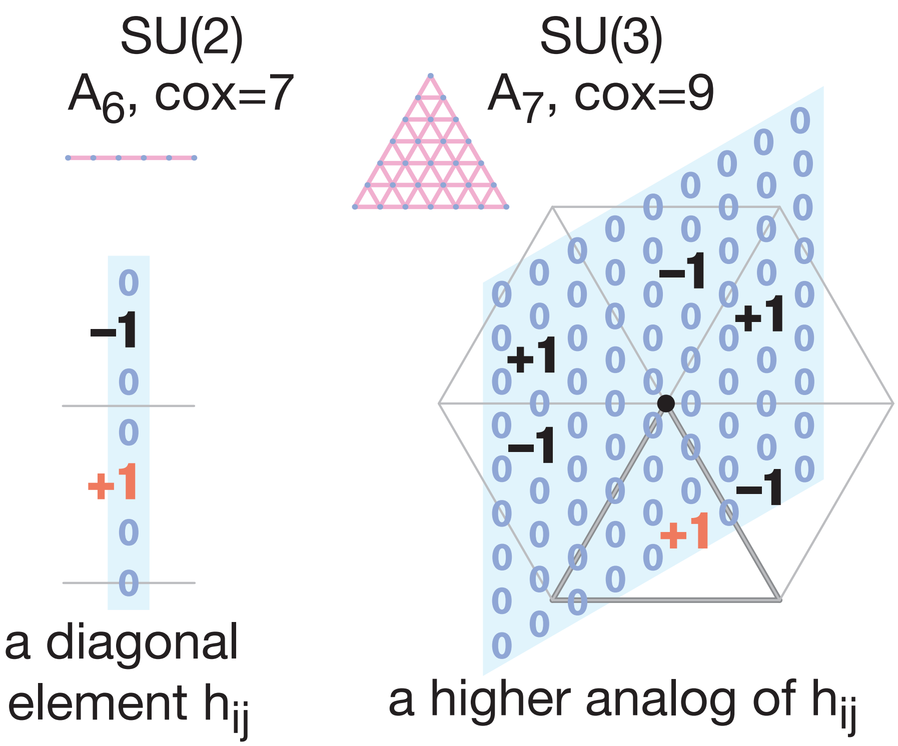
the generalized roots
form tetrahedra

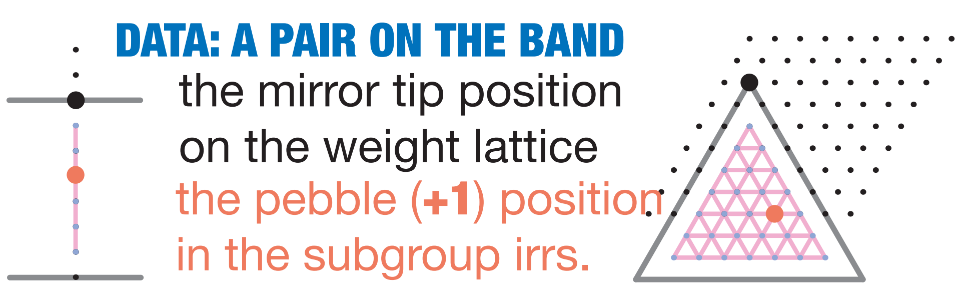


which suggest higher
composition laws.

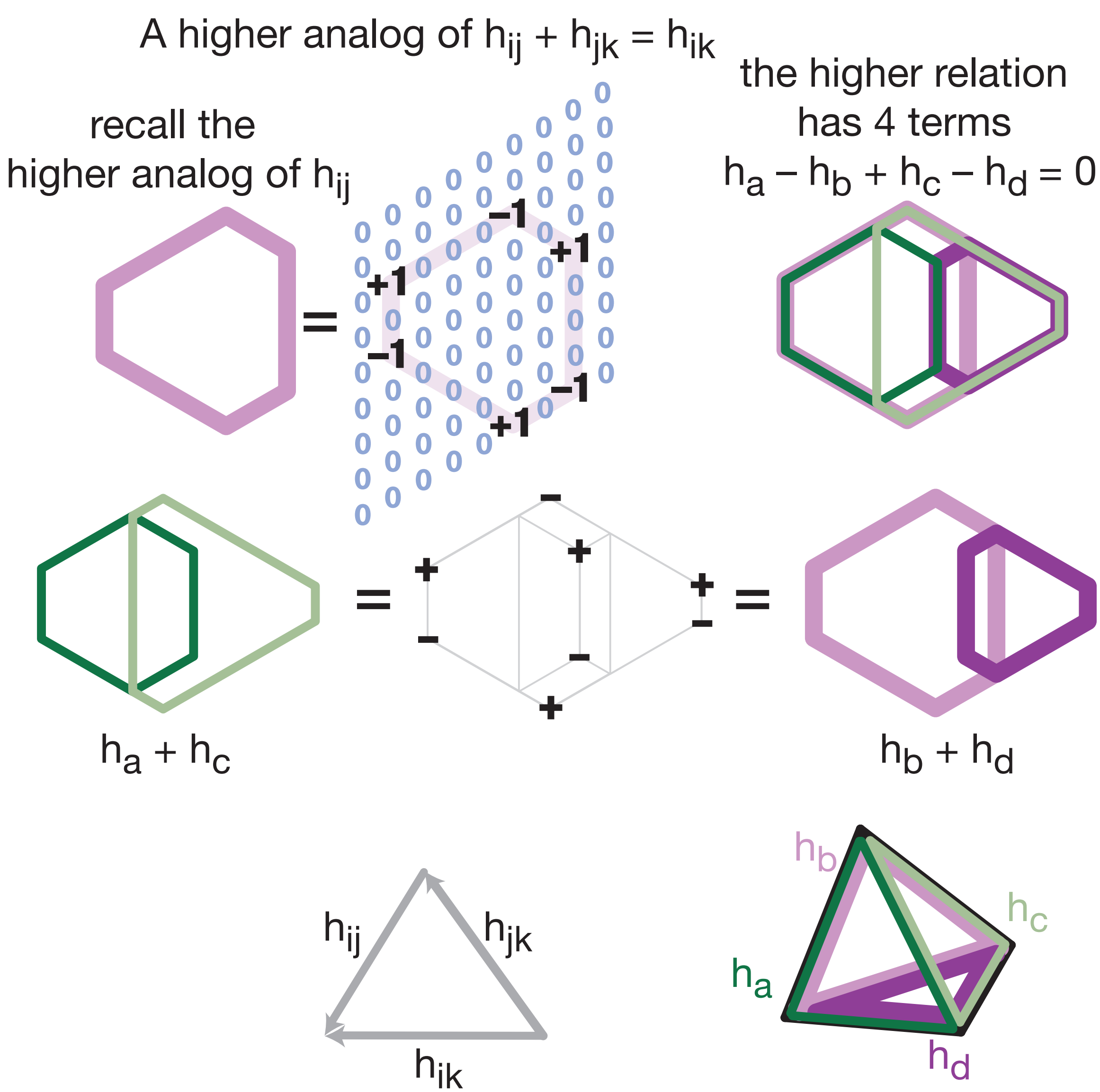
The roots obtained from $SU(K)_N$ suggest **higher analogs of simple Lie groups with K-nary composition laws**. These higher simple groups could be the base of
K-dimensional QFT.

• For the higher analogs of the A_n series the roots can be constructed as vectors in a manner analogous to the usual **diagonal matrices** $h_{ij} = e_{ii} - e_{jj}$.



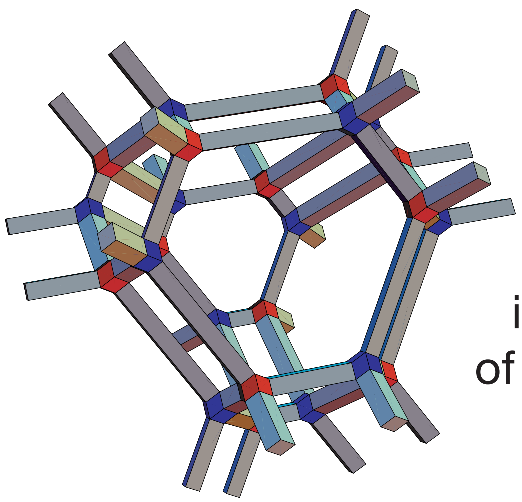


• The identity analogous to $h_{ij} + h_{jk} = h_{ik}$ has $K + 1$ terms for $SU(K)$.



• Here are the A_n series diagonal constructions for $SU(4)$.

A higher analog of h_{ij}

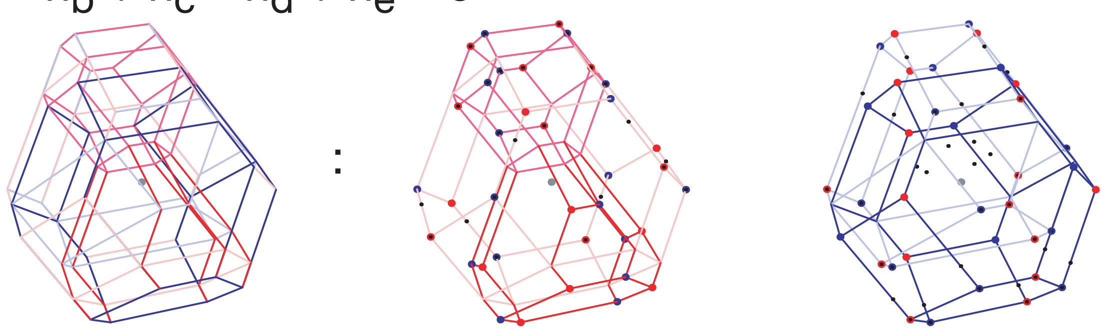


= +1
= −1
rest = 0

in an N x N x N period
of the root lattice of SU(4)

A higher analog of $h_{ij} + h_{jk} = h_{ik}$

$h_a - h_b + h_c - h_d + h_e = 0$

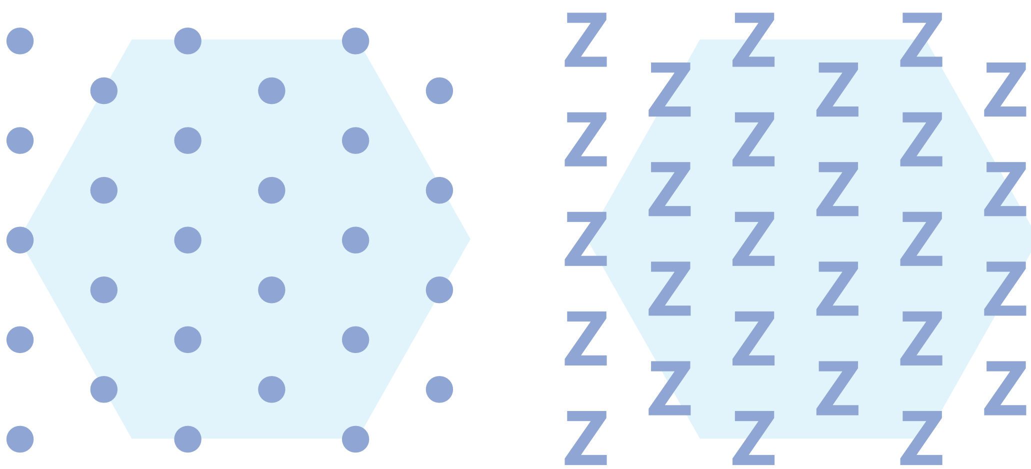


$h_a + h_c + h_e \quad = \quad h_b + h_d$

LATTICE QUANTIZATION: THE A_n SERIES

SU(2)

period 3, sum over period = 0
–> SU(3)



period 4, sum over period
in each Weyl direction = 0
–> $SU(3)_4$
(also: orbifolds, exceptional lattices)

Quantum Field Theory and Tensoriality.

• If $\mathcal{H}$ is the Hilbert space describing a particle (boson), then n bosons are described by the symmetric tensor power $1/n!\mathcal{H}^{\otimes_{\mathsf{s}} n}$. A magma, called quantum field theory (QFT), of continuously creating and annihilating bosons of the same kind is thus described by the symmetric space

$$\mathbf{S}\mathcal{H} = e_{\mathsf{s}}^{\mathcal{H}} = \bigoplus_{n=0}^{\infty} \frac{1}{n!}\mathcal{H}^{\otimes_{\mathsf{s}} n}.$$

These spaces behave **tensorially**,

$$e_{\mathsf{s}}^{\mathcal{H}\oplus\mathcal{K}} = e_{\mathsf{s}}^{\mathcal{H}} \otimes_{\mathsf{s}} e_{\mathsf{s}}^{\mathcal{K}}$$

but the spaces obtained this way are **too big**.

• One needs **smaller spaces** which behave **tensorially** (i.e. the sections over $U \cup V$ should be the **tensor product** – rather than the **direct sum** – of sections over U and V respectively) but which are not $e^{\mathcal{H}}$.

• Hom **spaces behave tensorially**. From objects with usual binary laws, such as Irr $SU(K)_N$, the QFT is 2-dimensional. This is the algebraic foundation of 2 dimensional conformal field theory and string theory.

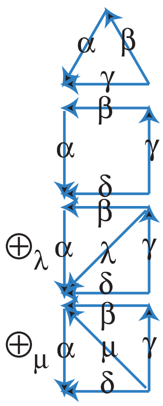
• For a realistic 4-dimensional space-time QFT, one needs Hom spaces of objects having **quaternary composition laws** (i.e. compose 4 objects to get a 5-th one).

• The **higher associativity** required is dictated by the **topological structure of 4-dimensional space**.

ALGEBRAIC DATA:

Irr (objects: e.g. Irr G for G (quantum) group,
or irreducible bimodules{$_A X_A$})

FUNCTOR



$\stackrel{H}{\Rightarrow}$ Hom[α⊗β,γ]   α,β,γ ∈ Irr

$\stackrel{H}{\Rightarrow}$ Hom[α⊗β⊗γ,δ]

|||

$\stackrel{H}{\Rightarrow}$ $\oplus_\lambda$Hom[α⊗β,λ]⊗Hom[λ⊗γ,δ]

||| associativity 

$\stackrel{H}{\Rightarrow}$ $\oplus_\mu$Hom[β⊗γ,μ]⊗Hom[α⊗μ,δ]

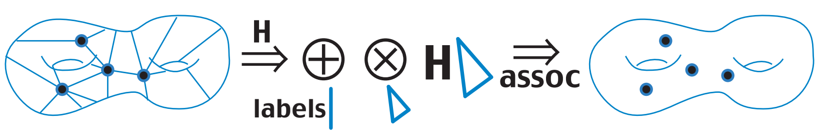
↑TENSORIAL: 2d QFT

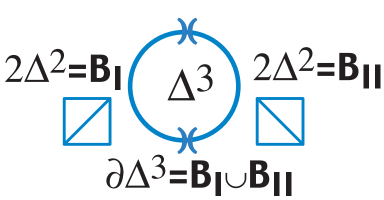
TOPOLOGICAL QUANTUM FIELD THEORY

 ⇒ 3d TQFT: topological invariants of (empty) 3-manifolds, knots by triangulation

(associativity coefficients)

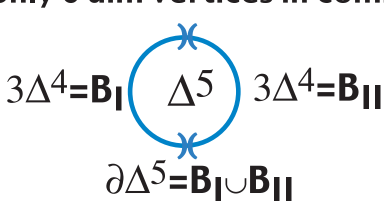
QUANTUM FIELD THEORY



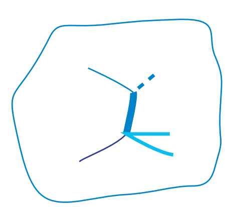


⇒ 2d QFT: tensorial Hilbert space for 2d space (= 1 space + 1 time) with 0d particles

2d associativity
(only 0 dim vertices in common)



⇒



4d associativity
(only 1 dim edges in common)

4d QFT: tensorial Hilbert space for 4d space (= 3 space + 1 time) with 1d particles (Feynman diagrams)

Here are some highlights of the construction and classification of quantum subgroups.

Modular group representaions

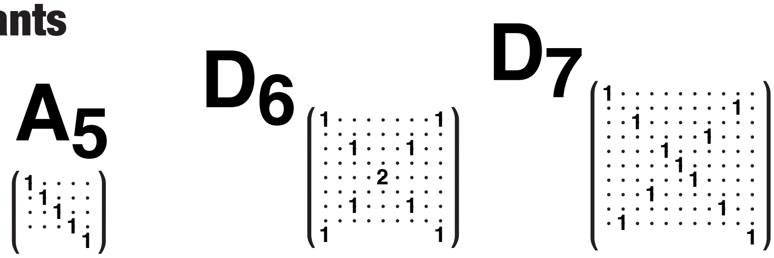
An N-dimensional complex representation of SL(2,Z) associated to $SU(2)_q$ at $q^N=1$
Hurwitz,(... 150 yrs later rediscovered in physics) Verlinde

$$\rho(S)= (\sin((k+1)(l+1)\pi/N)_{kl=0,..,N-1}$$

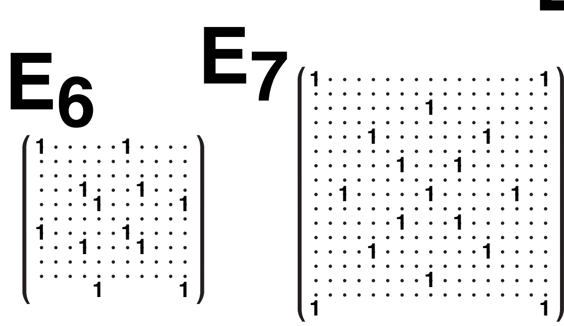
$$\rho(T)= (\delta_{kl} \exp((k+1)^2\pi/(2N))_{kl=0,..,N-1}$$

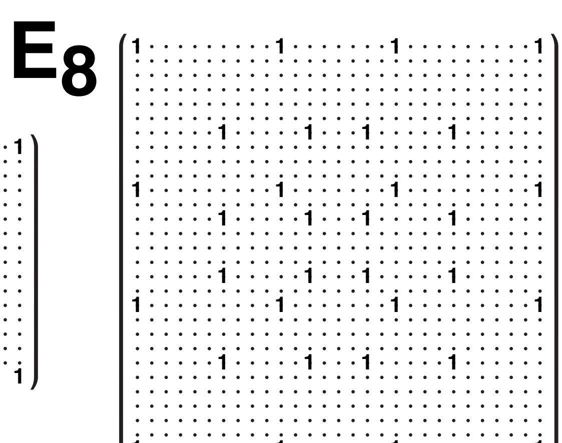
The representation above is NOT irreducible.
A modular invariant matrix (M_{kl}) is an intertwiner with entries M_{kl} natural numbers and $M_{00}=1$.
These were classified by Itzykson, Capelli, Zuber (1982)
Modular invariants corresponding to $SU(3)_q$ classified by T. Gannon

Modular invariants
series A_n, D_n



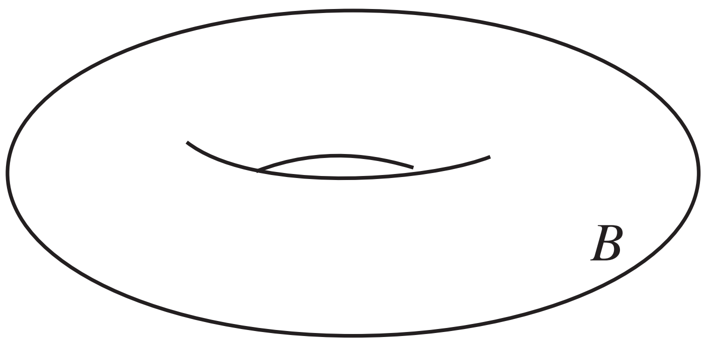
exceptionals E_6, E_7, E_8



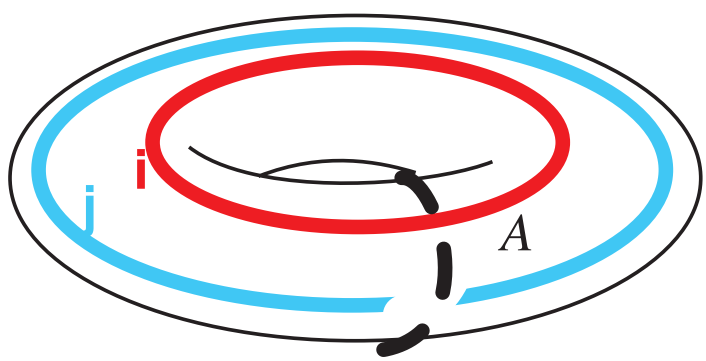


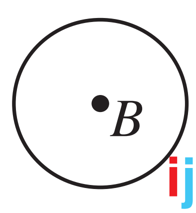
The diagonal terms: exponents of the A, D, E graph
The off diagonal terms were a mistery before our theory
-explained by the structure of quantum subgroups of SU(2)

The modular invariant

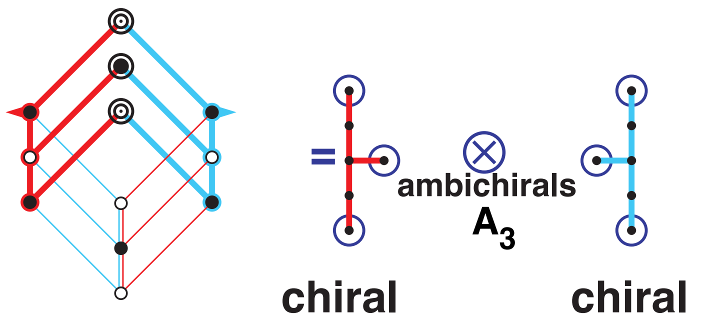


$=$

$= \sum_{ij} M_{ij}^{(B)}$ 

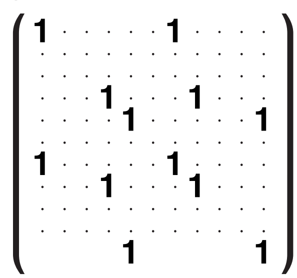
with $M_{ij}^{(B)} = \dim$ 

The quantum self-symmetries of a graph (e.g. E_6)
– the internal structure of the boundary data -



E_6-E_6 chiral left E_6 chiral right E_6

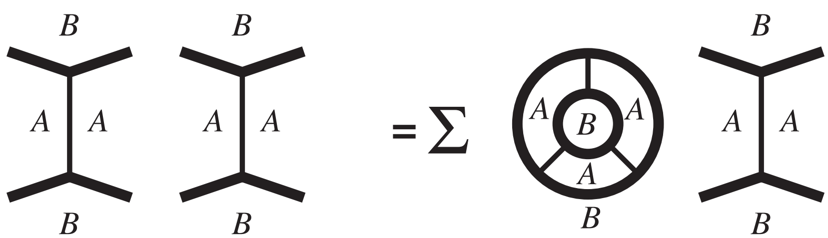
Explain the modular invariant:



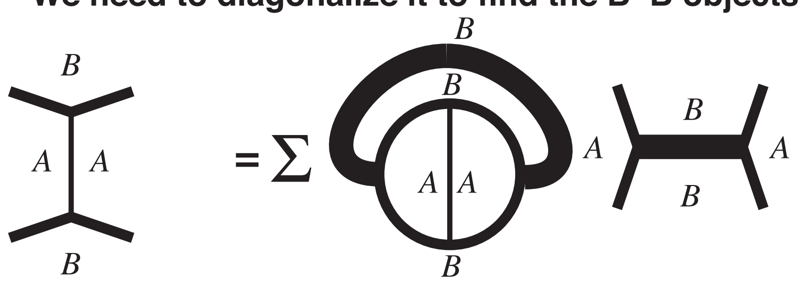
- each modular block is an ambichiral
- entries count paths on joint graph
- the first line modular invariants M_{0k} count the Kleinian invariants of the chiral graph
- M_{0k} also describe the characters of the matrix of the chiral graph
- the modular invariants M_{kl} count the Kleinian invariants and characters of the total graph
- the diagonal invariants M_{kk} describe the characters of the matrix of the module graph (as observed by Zuber)

The double triangle boundary Hopf algebra

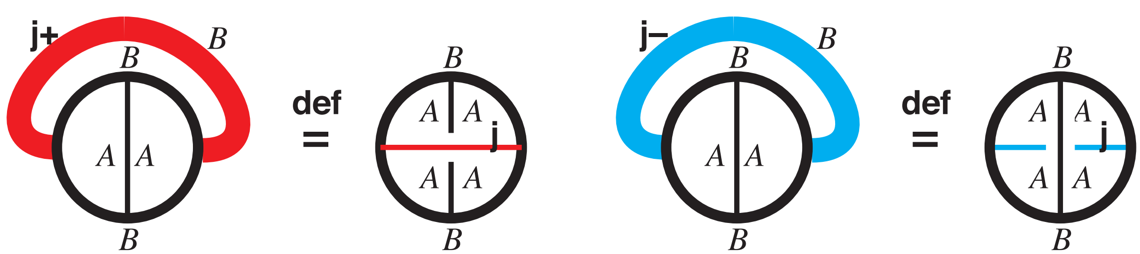
The A-B objects can be properly studied only if the much richer B-B objects can be understood.



We need to diagonalize it to find the B–B objects

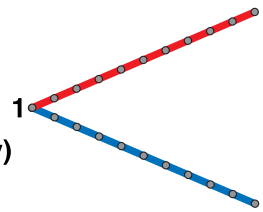


Very simple idea: **use braiding to define some B–B objects, chiral + and chiral –,** in terms of known A–A objects

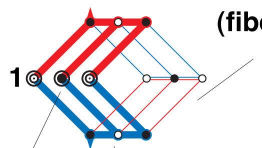


Ideally the chiral + and chiral – objects would be irreducible and yield everything

B–B:
(ideally)



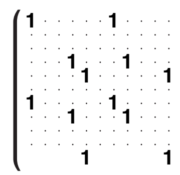
B–B:
(in fact)



all = span of + and –
(fibered product of chirals
over ambichirals)

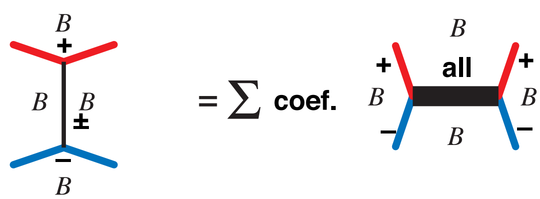
both + and –
(ambichiral)

branching
(Kleinian invariants
by Schur's lemma)



All these phenomena are read in the modular matrix,
which also gives the characters of all the graphs

Finally, a second Hopf algebra



shows that all B–B bimodules arise as a Hopf algebra product of the chiral + and chiral – subsystems, fibered over the ambichirals.