Robots, Trade, and Luddism *

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Abstract

Technological change, from the advent of robots to expanded trade opportunities, tends to create winners and losers. When are such changes welcome? How should government policy respond? We consider these questions in a second best world, with a restricted set of tax instruments. We establish a number of new optimal tax formulas as well as bounds on those optimal taxes. While distributional concerns create a rationale for non-zero taxes on robots and trade, we show that more robots, more trade, and more inequality may be optimally met with lower taxes. In spite of tax tools being restricted, we also show that productivity improvements are always welcome and valued in the same way as in a first best world.

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1 Introduction

Robots and artificial intelligence technologies are on the rise. So are imports from China and other developing countries. Regardless of its origins, technological change creates opportunities for some workers, destroys opportunities for others, and generates significant distributional consequences, as documented in the recent empirical work of Autor, Dorn and Hanson (2013) and Acemoglu and Restrepo (2017) for the United States.

When should technological change be welcome? Should any policy response be in place? And if so, how should we manage new technologies? Should we become more luddites as machines become more efficient or more protectionist as trade opportunities expand? The goal of this paper is to provide a general second-best framework to help address these and other related questions.

In seeking answers to these questions one first needs to take a stand on the range of available policy instruments. Obviously, if the idilic lump-sum transfers are available, distribution can be done efficiently, without distorting production. Even in the absence of lump-sum transfers, if linear taxes are available on all goods and factors, production efficiency may hold, as in Diamond and Mirrlees (1971b). In both cases, zero taxes on robots and free trade are optimal. At another extreme, in the absence of any policy instrument, whenever technological progress creates at least one loser, a welfare criterion must be consulted and the status quo may be preferred.

Here, we focus on intermediate, and arguably more realistic, scenarios where tax instruments are available, but are more limited than those ensuring production efficiency. Our framework is designed to capture general forms of technological change. We consider two sets of technologies, which we refer to as "old" and "new". For instance, firms using the new technology may be producers of robots or traders that export some goods in exchange for others. Since we are interested in the optimal regulation of the new technology, we do not impose any restriction on the taxation of firms using that technology, e.g. taxes on robots or trade. In contrast, to allow for a meaningful trade-off between redistribution and efficiency, we restrict the set of taxes that can be imposed on firms using the old technology as well as on consumers and workers. In the economic environment that we consider, the after-tax wage structure can be influenced by tax policy, but not completely controlled.

We first characterize the structure of optimal taxes on old and new technology firms in general environments when non-linear income taxation is allowed, as in Mirrlees (1971). The work of Naito (1999) has proven that governments seeking to redistribute income from high- to low-skill workers may have incentives to depart from production efficiency.

Doing so manipulates relative wages, which cannot be taxed directly, and relaxes incentive compatibility constraints. Our analysis generalizes this result and and goes beyond qualiative insights by deriving optimal tax formulas, expressed in terms of shares and elasticities, as well as bounds on any Pareto efficient tax.

Section 4 turns to a specific example. We consider an economy with only one final good that can be used either for consumption or for producing robots. Workers and robots, in turn, are combined to produce the final good. To provide further intuition about the structure of optimal taxes, we set up this example with limited general equilibrium effects. Namely, we impose restrictions on technology so that robots may directly affect inequality by affecting relative marginal products of labor, but not indirectly through further changes in relative labor supply, as in Stiglitz (1982).

In the context of this economy, we provide a sharp characterization of the optimal tax on robots as a function of observables: factor shares, elasticities, and marginal income tax rates. Crucially, our formula does not require knowledge, or assumptions, about the Pareto weights assigned to different agents in societies. Those are implicitly revealed by the observed income tax schedule. We also use the additional structure of this economy to conduct comparative static analysis. In contrast to many popular discussions, we show that an increase in the productivity of robot producers may be associated with a higher share of robots in the economy, a rise in inequality, as reflected in Pareto distributions of earnings with fatter tails, but lower optimal taxes on robots.

Section 5 goes back to our general environment to study the welfare impact of new technologies under the assumption that constrained, but optimal policies are in place. Our key finding is a novel envelope result. It generalizes the evaluation of productivity shocks in first-best environments, as in Solow (1957) and Hulten (1978), to distorted economies. Because of restrictions on the set of available tax instruments, marginal rates of substitution may not be equalized across agents and marginal rates of transformation may not be equalized between new and old technology firms. Yet, "Immiserizing Growth," as in Bhagwati (1958), never arises. Provided that governments can tax new technology firms, the welfare impact of technological progress can be measured in the exact same way as in first best environments (despite not being first best).

In the case of terms-of-trade shocks, this implies that such shocks are beneficial if and only if they raise the value of the trade balance at current quantities. This also implies that despite the concerns for redistribution, and the restrictions on tax instruments, the gains from international trade can still be computed by integrating below the demand curves for foreign goods. The previous considerations may affect how much we trade, but not the mapping between observed trade flows and welfare, as in Arkolakis, Costinot and

Rodríguez-Clare (2012).

We conclude by discussing the implications of our envelope result for innovation. To put it in concrete terms, if robots tend to have a disproportionate effect on the wages of less skilled workers, and we care about redistribution, should we offer less grants to AI research that is likely to increase the supply of robots and increase inequality? Another direct implication of our envelope result is that the answer to the previous question is no.

Our paper makes three distinct contributions to the existing literature. The first one is a general characterization of the structure of optimal production taxes in environments with restricted factor income taxation. In so doing, we fill a gap between the general analysis of Diamond and Mirrlees (1971b), Diamond and Mirrlees (1971a), and Dixit and Norman (1980), which assumes that linear taxes on all factors are available, and specific examples, typically with two goods and two factors, in which only income taxation is available, as in the original work of Naito (1999), and subsequent work by Guesnerie (1998), Spector (2001), and Naito (2006).[1] On the broad spectrum of restrictions on available policy instruments, one can also view our analysis as an intermediate step between the work of Diamond and Mirrlees (1971b), Diamond and Mirrlees (1971a), and Dixit and Norman (1980) and the trade policy literature, as reviewed for instance in Rodrik (1995), where it is common to assume that the only instruments available for redistribution are trade taxes. Grossman and Helpman (1994) is a well-known example. We come back to this point in Section 3.

Our second contribution is a more specific analysis of the optimal tax on robots. In recent work, Guerreiro, Rebelo and Teles (2017) have studied a model with both skilled and unskilled workers as well as robots. Under the assumption that factor-specific taxes are unavailable, they find that a non-zero robot tax is, in general, optimal, in line with the work of Naito (1999). In the limit, however, if the productivity of robots is large enough, unskilled workers drop out of the workforce, at which point their wage relative to skilled workers is no longer manipulable, and the optimal tax on robots becomes zero. Although we share the same rationale for non-zero taxes on robots, our paper goes beyond signing the tax on robots by offering optimal tax formulas, that can be implemented using a few sufficient statics, as well as comparative static predictions relating technological progress to the magnitude of the tax.[2]

[1]In all three papers, like in Dixit and Norman (1980), the new technology is international trade. In another related trade application, Feenstra and Lewis (1994) study an environment where governments cannot subject different worker types to different taxes, but can offer subsidies to workers moving from one industry to another in response to trade. They provide conditions under which such a trade adjustment assistance program are sufficient to guarantee Pareto gains from trade, as in Dixit and Norman (1980).

[2]Our optimal tax formulas in Section 4 are also related to the work of Jacobs (2015) who considers in an alternative environment, without robots, but with the same tax instruments as in Naito (1999), a continuum

Our third contribution is a new perspective on the welfare impact of technological progress in the presence of distortions. In a first best world, the impact of small productivity shocks can be evaluated, absent any restriction on preferences and technology, using a simple envelope argument as in Solow (1957) and Hulten (1978). With distortions, evaluating the welfare impact of productivity shocks, in general, requires additional information about whether such shocks aggravate or alleviate underlying distortions. In an environment with markups, for instance, this boils down to whether employment is reallocated towards goods with higher or lower markups, as in Basu and Fernald (2002), Arkolakis, Costinot, Donaldson and Rodríguez-Clare (Forthcoming), and Baeqee and Farhi (2017). If the aggravation of distortions is large enough, technological progress may even lower welfare, as discussed by Bhagwati (1971). Here, we follow a different approach. Our analysis builds on the idea that while economies may be distorted and tax instruments may be limited, the government may still have access to policy instruments to control the new technology. If so, the envelope results of Solow (1957) and Hulten (1978) still hold, with direct implications for the measurement of the welfare gains from globalization and automation as well as for the taxation of innovation.

2 General Framework

Consider an economy comprising many goods, indexed by $i = 1, ..., N$, and a continuum of agents, indexed by their ability $\theta \in [\underline{\theta}, \bar{\theta}]$. F denotes the cumulative distribution of abilities in the population and f denotes the associated density function.

2.1 Agents

All agents have the same weakly separable preferences over goods and labor. The associated utility function is given by

$$U = u(C, n),$$
$$C = v(\{c_i\}),$$

where C denotes the sub-utility derived from the consumption of all goods, $\{c_i\}$, and n denotes labor supply. Both u and v satisfy standard regularity conditions. All agents face the same good prices, $\{q_i\}$, but may receive a different wage, $w(\theta)$, and be subject to different income taxes, $T(w(\theta)n; \theta)$. In what follows, we let $R(w(\theta)n; \theta) \equiv w(\theta)n -$

of workers, and no general equilibrium effects.

$T(w(\theta)n;\theta)$ denote the after-tax income of a worker of type θ who supplies n and earns $w(\theta)n$ before taxes. Though we will be primarily interested in the case of a unique income tax schedule, $R(w(\theta)n;\theta) \equiv R(w(\theta)n)$, this extra generality will be helpful to relate our analysis to earlier work in the literature.

For our purposes, it will prove convenient to start with the expenditure minimization problem of the agents. Given the weak separability of agents' preferences, the optimal consumption and labor decisions can be solved for in two steps. For given good prices, $\{q_i\}$, and a given level of aggregate consumption, $C(\theta)$, the optimal consumption vector, $\{c_i(\theta)\}$, solves the lower-level expenditure minimization problem,

$$\{c_i(\theta)\} \in \text{argmin}_{\{c_i\}}\{\sum q_i c_i | v(\{c_i\}) \geq C(\theta)\}, \tag{1}$$

In turn, the optimal aggregate consumption, $C(\theta)$, and labor supply, $n(\theta)$, solve the upper-level problem,

$$C(\theta), n(\theta) \in \text{argmin}_{C,n}\{e(\{q_i\}, C) - R(w(\theta)n;\theta) | u(C,n) \geq U(\theta)\}, \tag{2}$$

where $e(\{q_i\}, C) \equiv \min_{\{c_i\}}\{\sum q_i c_i | v(c) \geq C\}$ denotes the lower-level expenditure function and $U(\theta)$ is the utility level of agent θ. Finally, budget balance requires

$$e(\{q_i\}, C(\theta)) = R(w(\theta)n;\theta). \tag{3}$$

2.2 Firms

There are two types of firms, each with access to a different technology, which we refer to as "old" and "new," and each potentially facing different taxes.

Old Technology Old technology firms choose their output, $\{y_i\}$, and labor inputs, $\{n(\theta)\}$, in order to maximize their profits taking good prices, $\{p_i\}$, and the schedule of wages, $\{w(\theta)\}$, as given. Because of producer taxes or subsidies, $\{t_i\}$, producer prices may differ from consumer prices, $p_i = q_i/(1+t_i)$. The solution to the profit maximization problem of the old technology firms, $\{n(\theta), y_i\}$, is such that

$$\{y_i\}, \{n(\theta)\} \in \text{argmax}_{\{\tilde{y}_i\},\{\tilde{n}(\theta)\}}\{\sum p_i \tilde{y}_i - \int w(\theta)\tilde{n}(\theta)dF(\theta) | G(\{\tilde{y}_i\}, \{\tilde{n}(\theta)\}) \leq 0\}, \tag{4}$$

where G determines the production set of old technology firms. Throughout we assume constant returns to scale so that G is convex and homogeneous of degree one.

New Technology New technology firms solve a similar problem. They choose their output, $\{y_i^*\}$, in order to maximize their profits taking good prices, $\{p_i^*\}$, as given. Taxes on new technology firms, $\{t_i^*\}$, may differ from those on old ones, leading to $p_i^* = q_i/(1+t_i^*) \neq p_i$. The supply of new technology firms, y^*, is such that

$$\{y_i^*\} \in \arg\max_{\{\tilde{y}_i\}} \{\sum p_i^* \tilde{y}_i | G^*(\{\tilde{y}_i\};\phi) \leq 0\}, \tag{5}$$

where G^* is convex and homogeneous of degree one and ϕ is a productivity shock, which we will use to parametrize technological change. Throughout we assume that G^* is strictly decreasing in ϕ. Hence an increase in ϕ is a productivity improvement.

Beside the taxes that they face, new technology firms differ from old ones in that they can only transform some goods into others. Provided that the set of taxes $\{t_i^*\}$ is unrestricted, this implies that governments can fully control the decision of new technology firms, $\{y_i^*\}$, regardless of the restrictions that may affect commodity taxation, $\{t_i\}$, and labor taxation, T. This is the critical assumption upon which our analysis will build.

The two examples of new technology that we have in mind are international trade and the production of physical capital and machines, such as robots. In the first case, new technology firms may be traders with production set,

$$G^*(\{y_i^*\};\phi) = \sum \bar{p}_i(\phi) y_i^*,$$

where $\bar{p}_i(\phi)$ denotes the world price of good i. In this environment, an increase in ϕ corresponds to a positive terms-of-trade shock. In the second case, new technology firms may be robot-producers that transform a composite of all other goods in the economy, call it gross output, into robots. We study such an example in detail in Section 4.

2.3 Competitive Equilibrium

For all goods, demand is equal to supply. In vector notation, this can be expressed as

$$\int c_i(\theta) dF(\theta) = y_i + y_i^*. \tag{6}$$

We are now ready to describe a competitive equilibrium with non-linear income taxes, T, and linear producer taxes, $\{t_i\}$ and $\{t_i^*\}$. This corresponds to $\{c(\theta)\}$, $\{n(\theta)\}$, $\{C(\theta)\}$, $\{u(\theta)\}$, $\{y_i\}$, $\{y_i^*\}$, $\{w(\theta)\}$, $\{p_i\}$, $\{p_i^*\}$, and $\{q_i\}$ such that: (i) agents and firms behave optimally, conditions (1)-(5); (ii) good markets clear, condition (6); and (iii) good prices satisfy the two following non-arbitrage conditions: $p_i = q_i/(1+t_i)$ and $p_i^* = q_i/(1+t_i^*)$

for all i. Note that since all markets clear and budget constraints hold, the government's budget constraint must hold as well, an expression of Walras' law. For the same reason, we can normalize good prices and taxes such that $p_1 = p_1^* = q_1 = 1$ and $t_1 = t_1^* = 0$.

To describe competitive equilibria in a compact manner and prepare the description of the government's problem, we introduce the following notation. On the demand side, we let $c_j(\{q_i\}, C(\theta))$ denote the consumption of good j that solves (1) given consumer prices, $\{q_i\}$, and aggregate consumption, $C(\theta)$; we let $C(n(\theta), U(\theta))$ denote the aggregate consumption required to achieve utility $U(\theta)$ given labor supply $n(\theta)$, that is the solution to $u(C, n(\theta)) = U(\theta)$; and we let $c_j(\{q_i\}, \{n(\theta)\}, \{U(\theta)\}) = \int c_j(\{q_i\}, C(n(\theta), U(\theta)))dF(\theta)$ denote the total demand for good j. Likewise, on the supply side, we let $y_j(\{p_i\}, \{n(\theta)\})$ denote the output of good j that solves (4) given prices, $\{p_i\}$, and labor demand, $\{n(\theta)\}$, and we let $w(\{p_i\}, \{n(\theta)\}; \theta)$ denote the associated equilibrium wage for each agent θ.[3]

Note that in general, both the demand of the agents, $c_j(\{q_i\}, \{n(\theta)\}, \{U(\theta)\})$, and the supply of old technology firms, $y_j(\{p_i\}, \{n(\theta)\})$, may be sets. This happens whenever agents and firms operate on flat portions of their indifference curves and production possibility frontiers, respectively. Though our results generalize to environments where such situations may arise, we will ignore them for expositional purposes.

2.4 The Government's Problem

The problem of the government is to choose a competitive equilibrium with an income tax schedule, T, and producer taxes on old and new technology firms, $\{t_i\}$ and $\{t_i^*\}$, in order to maximize

$$\int U(\theta) d\Lambda(\theta),$$

where Λ denotes the distribution of Pareto weights. Λ is positive, increasing, right-continuous, and normalized so that $\Lambda(\bar{\theta}) = 1$. The utilitarian benchmark corresponds to $\Lambda = F$. The Rawlsian benchmark corresponds to $\Lambda(\theta) = 1$ for all θ, that is, full weight at $\underline{\theta}$. [4]

Throughout our analysis, we assume that taxes on new technology firms are unrestricted, $t_i^* \in [-1, \infty)$ for all i. Hence, the government can freely choose the vector of

[3]The first-order conditions associated with (4) imply

$$w(\{p_i\}, \{n(\theta)\}; \theta) = \frac{G_{n(\theta)}(\{y_j(\{p_i\}, \{n(\theta)\})\}, \{n(\theta)\}) \times \sum p_j y_j(\{p_i\}, \{n(\theta)\})}{\int n(\theta') G_{n(\theta')}(\{y_j(\{p_i\}, \{n(\theta)\})\}, \{n(\theta)\}) dF(\theta')},$$

with $G_{n(\theta)} \equiv \partial G / \partial n(\theta)$.

[4]Pareto weights may themselves derive from political-economy considerations, like the agents' heterogeneous ability to make political contributions, as in Grossman and Helpman (1994).

prices faced by new technology firms, $\{p_i^*\}$, irrespectively of what the prices faced by old technology firms and agents, $\{p_i, q_i\}$, may be. In contrast, we assume that the government may be limited both in its ability to tax labor income, through T, and old technology firms, through $\{t_i\}$. In what follows, we let $\mathcal{R}$ denote the set of feasible after-tax schedules, i.e. $\mathcal{R} \equiv \{R(\cdot;\cdot)|R(x;\theta) = x - T(x;\theta)$ for all x and θ for some feasible $T(\cdot;\cdot)\}$, and we let $\mathcal{P}$ denote the set of feasible prices for agents and old technology firms, i.e. $\mathcal{P} \equiv \{\{p_i, q_i\}|p_i = q_i/(1+t_i)$ for all i for some feasible $\{t_i\}\}$. We discuss a number of examples in the next section.

Using the previous notation, the government's problem can be expressed as

$$\max_{\{U(\theta)\},\{n(\theta)\},R\in\mathcal{R},\{p_i,q_i\}\in\mathcal{P}} \int U(\theta)d\Lambda(\theta) \tag{7}$$

subject to

$$n(\theta), U(\theta) \in \operatorname{argmax}_{n,U}\{U|e(\{q_i\}, C(n,U)) = R(w(\{p_i\},\{n(\theta)\};\theta)n;\theta)\},$$

$$G^*(\{c_j(\{q_i\},\{n(\theta)\},\{U(\theta)\}) - y_j(\{p_i\},\{n(\theta)\})\};\phi) \leq 0.$$

The first constraint is the counterpart to the agents' upper-level optimality conditions (2) and (3), whereas the second constraint is the counterpart to the good market clearing condition (6). The lower-level optimality condition (1) is already captured by the total demand, $c_j(\{q_i\},\{n(\theta)\},\{U(\theta)\})$. Likewise, the profit-maximization of old technology firms, condition (4), is already captured by the supply and wage schedules, $y_j(\{p_i\},\{n(\theta)\})$ and $w(\{p_i\},\{n(\theta)\};\theta)$. Finally, since the taxes on new technology firms $\{t_i^*\}$ are unrestricted, they can always be chosen such that the profit-maximization of new technology firms, condition (5), holds for any feasible vector of output, $\{y_i^*\}$.[5]

3 Managing New Technologies

3.1 Setting the Stage

New technologies improve efficiency, but may have adverse distributional consequences. Depending on the availability of tax instruments, governments may choose different strategies to manage these technologies. To set the stage for our formal analysis, we

[5]This is true regardless of whether $\{y_i^*\}$ is on the production possibility frontier of new technology firms. If it is not, taxes on new firms should simply be set to $t_i^* = \infty$ for all i. We shall ignore this knife-edge case.

briefly discuss how important benchmark results map into the government's problem of Section 2.4 and how we will depart from them.

Unrestricted Taxation Consider first the extreme case where all tax instruments are unrestricted. This implies, in particular, that agent-specific lump-sum transfers are available, like in the Second Welfare Theorem. In this case, the government's problem reduces to

$$\max_{\{U(\theta)\},\{n(\theta)\},\{p_i,q_i\}} \int U(\theta) d\Lambda(\theta)$$

subject to

$$G^*(\{c_j(\{q_i\},\{n(\theta)\},\{U(\theta)\}) - y_j(\{p_i\},\{n(\theta)\})\};\phi) \leq 0.$$

In matrix notation, the associated first-order conditions with respect to $\{p_i\}$ and $\{q_i\}$ are given by

$$D_p y \cdot \nabla_{y^*} G^* = 0;$$
$$D_q c \cdot \nabla_{y^*} G^* = 0,$$

where $D_p y \equiv \{dy_j/dp_i\}$ and $D_q c \equiv \{dc_i/dq_j\}$ are $N \times N$ matrices and $\nabla_{y^*} G^* = \{dG^*/dy_j^*\}$ is a $N \times 1$ vector.

To go from the previous first-order conditions to the optimal price gaps, note that profit maximization by new technology firms requires marginal rates of transformation to be equal to the relative prices that they face. Thus, the vector of new prices, $p^* \equiv \{p_i^*\}$, must be collinear with $\nabla_{y^*} G^*$, and, in turn, we must have $D_p y \cdot p^* = D_q c \cdot p^* = 0$. Likewise, profit maximization by old technology requires $D_p y \cdot p = 0$—changes in output can only have second-order effects on firms' revenues—whereas expenditure minimization by agents requires $D_q c \cdot q = 0$—changes in consumption can only have second-order effects on agents' expenditure. Combining the previous observations, we get

$$D_p y \cdot (p - p^*) = 0,$$
$$D_q c \cdot (q - p^*) = 0.$$

Under our normalization, $p_1 = p_1^* = q_1 = 1$, this implies that all prices, p, p^*, and q should be equal and, accordingly, that all good taxes should be zero. Not surprisingly, if lump-sum transfers are available, there is no trade-off between efficiency and redistribution.

Unrestricted Linear Taxation Suppose now that $\{t_i\}$ and $\{t_i^*\}$ remain unrestricted, but that labor taxation is restricted to linear taxes, $R(w(\theta)n;\theta) = (1 - T(\theta))w(\theta)n$. This is the case of unrestricted linear taxation considered by Diamond and Mirrlees (1971b), Diamond and Mirrlees (1971b), and Dixit and Norman (1986). Let $r(\theta) \equiv (1 - T(\theta))w(\theta)$ denote the wage schedule faced by workers. In this situation, we can express the government's problem as

$$\max_{\{U(\theta)\},\{n(\theta)\},\{p_i,q_i\},\{r(\theta)\}} \int U(\theta)d\Lambda(\theta)$$

subject to

$$n(\theta), U(\theta) \in \text{argmax}_{n,U}\{U|e(\{q_i\}, C(n,U)) = r(\theta)n\},$$

$$G^*(\{c_j(\{q_i\}, \{n(\theta)\}, \{U(\theta)\}) - y_j(\{p_i\}, \{n(\theta)\})\};\phi) \leq 0.$$

Since $\{p_i\}$ only appears in the good market clearing condition, the associated first-order condition is unchanged,

$$D_p y \cdot \nabla_{y^*} G^* = 0.$$

For the same reason as before, we therefore still have the equality of p and p^*. This is Diamond and Mirrlees's (1971b) production efficiency result: there should be no differences between the taxes faced by old and new technology firms. In a trade context, this implies that the solution to the government's problem may feature consumer taxes, $q \neq p = p^*$, but not trade taxes. A corollary of this observation is that the government can always find an allocation with consumer taxes, $t \neq 0$, and no trade taxes, $t^* = 0$ that leads to higher welfare than the autarky equilibrium, i.e, an equilibrium with prohibitive trade taxes. This is Dixit and Norman's (1986) result.

An Intermediate Case The assumption that lump-sum transfers are available or that all factors can be taxed at a different rate are clearly strong ones. In the trade policy literature, as reviewed for instance in Rodrik (1995), it is common to make the other extreme assumption that the only instruments available for redistribution are trade taxes. In this paper, we wish to explore further the intermediate, and more realistic, case where lump-sum transfers and factor taxation à la Diamond and Mirrlees (1971b) and Dixit and Norman (1986) are not available, but some form of income taxation may still be available and used for redistributive purposes.

3.2 The Government's Problem with Non-Linear Income Taxation

In the rest of this section and the next, we assume that factor-specific taxes are unavailable, but that income taxation is: $T(w(\theta)n;\theta) \equiv T(w(\theta)n)$ for all θ, with no constraint on the non-linear income tax schedules. For now, we remain agnostic about the set of feasible good prices for old technology firms and agents, $\mathcal{P}$.

In this intermediate case, the government's problem becomes

$$\max_{\{U(\theta)\},\{n(\theta)\},\{p_i,q_i\}\in\mathcal{P}} \int U(\theta)d\Lambda(\theta)$$

subject to

$$U(\theta) = \max_{\theta'} u\left(C(n(\theta'),U(\theta')), n(\theta')\frac{w(\{p_i\},\{n(\theta)\};\theta')}{w(\{p_i\},\{n(\theta)\},\theta)}\right),$$

$$G^*(\{c_j(\{q_i\},\{n(\theta)\},\{U(\theta)\}) - y_j(\{p_i\},\{n(\theta)\})\};\phi) \leq 0.$$

The first constraint is an incentive compatibility (IC) constraint ensuring that $\theta' = \theta$ is optimal.[6] Conversely, for any allocation $\{U(\theta), n(\theta)\}$ satisfying incentive compatibility, we can find a tax schedule, T, and a retention function, R, such that the constraint on the optimality of $n(\theta)$ in the original government's problem holds for all θ.

Let $u_n \equiv \partial u/\partial n$ denote the partial derivative of u with respect to n. The envelope condition associated with the IC constraint gives

$$U'(\theta) = -u_n(C(n(\theta),U(\theta)),n(\theta))n(\theta)\omega(\{p_i\},\{n(\theta)\};\theta)$$

where$\omega(\{p_i\},\{n(\theta)\};\theta)$ is a local measure of wage inequality,

$$\omega(\{p_i\},\{n(\theta)\};\theta) \equiv \frac{w_\theta(\{p_i\},\{n(\theta)\};\theta)}{w(\{p_i\},\{n(\theta)\};\theta)},$$

with $w_\theta \equiv \partial w/\partial\theta$. For piecewise differentiable allocations, the envelope condition and monotonicity of the mapping from wages, $w(\{p_i\},\{n(\theta)\};\theta)$, to before-tax earnings, $w(\{p_i\},\{n(\theta)\};\theta)n$ is equivalent to incentive compatibility. We will focus on cases where $w(\{p_i\},\{n(\theta)\};\theta)$ is increasing in θ, which for a given allocation can be interpreted as a normalization or

[6] In order to achieve the earnings $n(\theta')w(\{p_i\},\{n(\theta)\};\theta')$ of an agent of type θ', an agent of type θ must supply $n(\theta')w(\{p_i\},\{n(\theta)\};\theta')/w(\{p_i\},\{n(\theta)\},\theta)$ units of labor. Since all agents have the same weakly separable preferences, if agent θ mimics agent θ' and receive the same earnings, she must also achieve the same aggregate consumption, $C(n(\theta'),U(\theta'))$.

ordering of θ. Under the previous conditions, we can rearrange our planning problem as

$$\max_{\{U(\theta)\},\{n(\theta)\},\{p_i,q_i\}\in\mathcal{P}} \int U(\theta)d\Lambda(\theta) \tag{8a}$$

subject to

$$U'(\theta) = -u_n(C(n(\theta),U(\theta)),n(\theta))n(\theta)\omega(\{p_i\},\{n(\theta)\};\theta), \tag{8b}$$

$$G^*(\{c_j(\{q_i\},\{n(\theta)\},\{U(\theta)\}) - y_j(\{p_i\},\{n(\theta)\})\};\phi) \leq 0. \tag{8c}$$

3.3 Optimal Taxes on New and Old Technology Firms

We study separately the cases with and without taxes on old technology firms.

Case I: Taxes on old and new technology firms are available In this case, good prices $\{p_i, q_i\}$ are unrestricted. The first-order condition with respect to $\{p_i, q_i\}$ therefore implies

$$D_p y \cdot \nabla_{y^*} G^* = -[\int \mu(\theta)u_n(\theta)n(\theta)(\nabla_p \omega(\theta))d\theta]/\gamma,$$
$$D_q c \cdot \nabla_{y^*} G^* = 0,$$

where $\mu(\theta)$ denotes the Lagrange multiplier associated with agent θ's IC constraint, γ denotes the Lagrange multiplier associated with the good market clearing condition, $u_n(\theta) \equiv u_n(C(n(\theta),U(\theta)),n(\theta))$, and $\nabla_p\omega(\theta) \equiv \{d\omega(\{p_i\},\{n(\theta)\};\theta)/dp_j\}$. Like in Section 3.1, we can rearrange the previous expressions in terms of price gaps using the fact that firms maximize profits and agents minimize expenditure,

$$D_p y \cdot (p^* - p) = -[\int \mu(\theta)u_n(\theta)n(\theta)(\nabla_p \omega(\theta))d\theta]/\gamma, \tag{9}$$
$$D_q c \cdot (p^* - q) = 0. \tag{10}$$

For the same reason as in Section 3.1, equation (10) immediately implies that $p^* = q$ and, in turn, that there should be no taxes on new technology firms, $t_i^* = 0$ for all i. The implications of equation (9) for optimal taxation are more subtle. It includes the Lagrange multipliers associated with the incentive compatibility constraint, $\mu(\theta)$, and the good market clearing condition, γ. Those are neither primitives nor observables. As a first step towards operationalizing the previous results, we therefore propose to solve for $\mu(\theta)/\gamma$, as a function of the distribution of Pareto weights, Λ, which we take as primitives, and observables, such as marginal income tax rates, $\tau(\theta) \equiv T'(w(\theta)n(\theta))$, and earnings, $x(\theta) \equiv w(\theta)n(\theta)$.

To do so, we can use the first-order condition with respect to $\{U(\theta)\}$ and the fact that $u_n(\theta)/u_C(\theta) = -(1-\tau(\theta))w(\theta)$. As formally established in Appendix A, this leads to

$$D_p y \cdot (p^* - p) = \int \psi(\theta)(1-\tau(\theta))x(\theta)(\nabla_p \omega(\theta))d\theta, \tag{11}$$

where the weight $\psi(\theta)$ is such that

$$\begin{aligned}\psi(\theta) = &\int_{\theta}^{\bar{\theta}} \exp[-\int_{\theta}^{z} \rho(v)dx(v)]u_C(z)\zeta(z)dF(z) \\ &-\frac{\int_{\underline{\theta}}^{\bar{\theta}} \exp[-\int_{\underline{\theta}}^{z} \rho(v)dx(v)]u_C(z)\zeta(z)dF(z)}{\int_{\underline{\theta}}^{\bar{\theta}} \exp[-\int_{\underline{\theta}}^{z} \rho(v)dx(v)]u_C(z)d\Lambda(z)} \int_{\theta}^{\bar{\theta}} \exp[-\int_{\theta}^{z} \rho(v)dx(v)]u_C(z)d\Lambda(z),\end{aligned} \tag{12}$$

with $\rho(\theta) \equiv w(\theta)\frac{\partial(u_n/u_C)}{\partial C}$ the partial derivative, with respect to aggregate consumption, of the marginal rate of substitution between earnings and aggregate consumption, $\zeta(\theta) \equiv G^*_{y^*_1} \sum q_j \frac{dc_j(\{q_i\},C(n(\theta),U(\theta)))}{dU(\theta)}$ the inverse of agent θ's marginal utility of income, and $u_C(\theta) \equiv u_C(C(n(\theta),U(\theta)),n(\theta))$.

Noting that $p^*_i - q_i = -\frac{t^*_i}{1+t^*_i}q_i$ and $p_i - q_i = -\frac{t_i}{1+t_i}q_i$, we can summarize the implications of equations (10) and (11) for the optimal ad-valorem taxes on old and new technology firms as follows.

Proposition 1. *If taxes on new and old technology firms are available, optimal taxes on new technology firms, $\{t^*_i\}$, are zero and optimal taxes on old technology firms, $\{t_i\}$, are such that*

$$D_p y \cdot (\frac{tq}{1+t}) = \int \psi(\theta)(1-\tau(\theta))x(\theta)(\nabla_p \omega(\theta))dF(\theta),$$

where $\frac{tq}{1+t} \equiv \{\frac{t_i q_i}{1+t_i}\}$ is a $N \times 1$ vector and $\psi(\theta)$ is given by equation (12).

The fact that optimal taxes on new technology firms are zero is an expression of the targeting principle. Here, as in Naito (1999), the rationale for good taxation is to manipulate wages and this is best achieved by manipulating the prices of old technology firms that, together with labor supply, determine these wages.[7] Specifically, by affecting the

[7] Mayer and Riezman (1987) establish a similar result in a trade context with inelastic factor supply and no income taxation. If both producer and consumer taxes are available, they show that only the former should be used. This result, however, requires preferences to be homothetic, as discussed in Mayer and Riezman (1989). Our result does not require this restriction. This reflects the fact that we have access to non-linear income taxation, leading to the envelope condition (8b) rather than the budget constraint, $e(q,u(\theta)) = w(p,n,\theta)n(\theta)$, in our planning problem. We come back to this point in footnote 8.

prices faced by old technology firms, the government may lower inequality, as measured by a decrease in $\omega(\theta)$, and relax the incentive compatibility constraint of agent θ. The lower inequality is, the more costly it becomes for more skilled agents to mimic the behavior of less skilled agents, and hence the lower their informational rents. This creates a first-order welfare gains from taxes on old technology firms. Intuitively, it is cheaper to achieve redistribution through income taxation if earnings, before income taxes, are already more equal.

The key difference between Proposition 1 and the work of Naito (1999), beside greater generality, is the fact that our analysis goes beyond the first-order condition (9) by solving for the endogenous Lagrange multipliers, $\mu(\theta)/\gamma$, as a function of primitives and observables. Under the assumption that upper-level preferences are quasi-linear, $u(C, n) \equiv C - h(n)$, this provides a particularly simple optimal tax formula. In this case, since $u_C(\theta) = 1$, $\zeta(\theta) = 1$, and $\rho(\theta) = 0$ for all θ, we have

$$\psi(\theta) = \Lambda(\theta) - F(\theta). \tag{13}$$

This leads to the following corollary.

Corollary 1. *If taxes on new and old technology firms are available and upper-level preferences are quasi-linear, optimal taxes on old technology firms satisfy*

$$D_p y \cdot (\frac{tq}{1+t}) = \int (\Lambda(\theta) - F(\theta))(1 - \tau(\theta))x(\theta)(\nabla_p \omega(\theta))dF(\theta).$$

Case II: Only taxes on new technology firms are available So far, we have assumed that taxes on both old and new technology firms are available. We now briefly discuss the case where only the latter are available. This is equivalent to assuming that set of feasible prices, $\mathcal{P}$, is such that $p_i = q_i$ for all i. Under this additional constraint, the first-order condition with respect to $\{p_i\}$ simply becomes

$$(D_p y - D_q c) \cdot (p^* - p) = -\int \frac{\mu(\theta)}{\gamma} u_n(\theta) n(\theta)(\nabla_p \omega(\theta))d\theta,$$

The rest of the analysis is unchanged. In particular, equation (12) still holds. These two observations lead to our next proposition.

Proposition 2. *If only taxes on new technology firms are available, optimal taxes satisfy*

$$(D_q c - D_p y) \cdot (\frac{pt^*}{1+t^*}) = \int \psi(\theta)(1 - \tau(\theta))x(\theta)(\nabla_p \omega(\theta))dF(\theta),$$

where $\frac{pt^*}{1+t^*} \equiv \{\frac{p_i t_i^*}{1+t_i^*}\}$ *is a* $N \times 1$ *vector and* $\psi(\theta)$ *is given by equation (12).*

The economics is the same as in the case of Proposition 1. The only difference is that a change in p_i now distorts both consumption and output decisions. Hence, it is the change in output net of consumption, $D_p y - D_q c$, that matters for the optimal level of the taxes.

In the case of quasi-linear upper-level preferences, we still have $\psi(\theta) = \Lambda(\theta) - F(\theta)$, leading to the counterpart of Corollary 1.

Corollary 2. *If only taxes on new technology firms are available and upper-level preferences are quasi-linear, optimal taxes satisfy*

$$(D_q c - D_p y) \cdot (\frac{pt^*}{1+t^*}) = \int (\Lambda(\theta) - F(\theta))(1-\tau(\theta))x(\theta)(\nabla_p \omega(\theta))dF(\theta).$$

In a trade context, Corollary 2 implies that two key determinants of optimal tariffs are: (i) the difference between the Pareto weights of the government, Λ, and the utilitarian one, F; and (ii) the elasticity of import demand, as captured by $D_p(c-y)$. These are the same determinants found in the optimal tariff formula of Grossman and Helpman (1994), where $\Lambda(\theta)$ reflects whether agents are politically organized or not.

This should be intuitive. The key difference between the class of problems that we consider and those in the political economy of trade literature is that we allow for endogenous labor supply and income taxation. So far, however, we have not used the first-order condition with respect to $\{n(\theta)\}$, which explains the similarity between Proposition 2 and the existing trade literature.[8] Here, labor supply considerations are implicitly captured by the optimal marginal tax rates, $\tau(\theta)$. We explore them in detail in the next section.

[8] Formally, in the absence of income taxation and with inelastic factor supply, the government's problem of Section 2.4 reduces to

$$\max_{\{U(\theta)\},\{p_i,q_i\}\in\mathcal{P}} \int U(\theta)d\Lambda(\theta),$$

subject to

$$e(\{q_i\}, U(\theta)) = w(\{p_i\}, \{n(\theta)\}; \theta)n(\theta),$$

$$G^*(\{c_j(\{q_i\}, \{n(\theta)\}, \{U(\theta)\}) - y_j(\{p_i\}, \{n(\theta)\})\}; \phi) \leq 0.$$

Under the restrictions that $p_i = q_i$, the first-order condition with respect to $\{p_i\}$ is given by

$$D_p(c-y) \cdot \nabla G^*_{y^*} = \int \frac{\mu(\theta)}{\gamma}[\nabla_p e - (\nabla_p w)n(\theta)]d\theta \neq 0.$$

Taking the first-order condition with respect to $U(\theta)$, as we did above, one can further relate $\mu(\theta)/\gamma$ to the Pareto weights to obtain Grossman and Helpman's (1994) formula.

3.4 Correlations and Bounds

Correlations Since the rationale behind taxes on old and new technology firms is to redistribute from the rich to the poor, by lowering inequality and relaxing IC constraints, it seems natural to expect, on average, higher taxes on old technology firms, relative to new technology firms, on goods for which higher prices, p_i, are associated with more inequality. Our next result formalizes this intuition.

Since old technology firms maximize profits, $D_p y$ must be positive semi-definite. Likewise, since agents minimize expenditure, $D_q c$ must be negative semi-definite. For any vector of price gaps, $p^* - p$, we must therefore have

$$(p^* - p)' \cdot D_p y \cdot (p^* - p) \geq 0,$$
$$(p^* - p)' \cdot D_q c \cdot (p^* - p) \leq 0,$$

where $(p^* - p)'$ denotes the transpose of $(p^* - p)$. Using these two observations, we obtain the following corollary of Propositions 1 and 2.

Corollary 3. *Regardless of whether or not only taxes on new technology firms are available, optimal price gaps between new and old technology firms are such that*

$$(p^* - p)' \cdot \int \psi(\theta)(1 - \tau(\theta))x(\theta)(\nabla_p \omega(\theta))dF(\theta) \geq 0.$$

In words, old technology firms should tend to have lower prices, i.e., be taxed more relative to new technology firms, in sectors that tend to increase inequality the most, i.e., those for which $\int \psi(\theta)(1 - \tau(\theta))x(\theta)(\nabla_p \omega(\theta))dF(\theta)$ is high. If taxes on new and old technology firms are available, this will take the form of higher taxes, t_i, on old technology firms. If only the former taxes are available, this will take the form of lower taxes, t_i^*, on new technology firms.

Bounds Up to this point, all our results about the structure of optimal taxes requires knowledge of the government's Pareto weights, Λ. We conclude this section by providing bounds on optimal taxes that dispense with such information. We focus on the case with quasi-linear upper-level preferences discussed in Corollaries 1 and 2. This is the case that we will study further in the next section.[9]

[9] Similar bounds can be obtained in the case with weakly separable, but not necessarily additively separable preferences.

From equation (13) and the fact that $\Lambda(\theta) \in [0,1]$, we know that

$$-F(\theta) \leq \psi(\theta) \leq 1 - F(\theta).$$

Let $\Theta_i^+ \equiv \{\theta \in [\underline{\theta}, \bar{\theta}] | \omega_{p_i}(\theta) > 0\}$ and $\Theta_i^- \equiv \{\theta \in [\underline{\theta}, \bar{\theta}] | \omega_{p_i}(\theta) < 0\}$ denote the set of agents for which an increase in p_i locally raises and lowers inequality, respectively. Using the previous notation, we obtain the following corollary of Proposition 1.

Corollary 4. *If taxes on new and old technology firms are available, optimal taxes on old technology firms, $\{t_i\}$, are such that*

$$D_{p_i}y \cdot \left(\frac{tq}{1+t}\right) \leq \int_{\Theta_i^+} (1-\tau(\theta))x(\theta)\omega_{p_i}(\theta)dF(\theta) - \int F(\theta)(1-\tau(\theta))x(\theta)\omega_{p_i}(\theta)dF(\theta),$$
$$D_{p_i}y \cdot \left(\frac{tq}{1+t}\right) \geq \int_{\Theta_i^-} (1-\tau(\theta))x(\theta)\omega_{p_i}(\theta)dF(\theta) - \int F(\theta)(1-\tau(\theta))x(\theta)\omega_{p_i}(\theta)dF(\theta),$$

where $D_{p_i}y$ denotes the i-th row of $D_p y$.

Beside the distortionary impact of good taxation on output, measured by $D_{p_i}y$, Corollary 4 implies that the only information required to bound the optimal taxes on old technology firms are data on, or estimates of, the impact of good prices on inequality, $\omega_{p_i}(\theta)$, earnings, $x(\theta)$, marginal income tax rates, $\tau(\theta)$, and the distribution of skills, $F(\theta)$.[10] Note also that for any good i that raises inequality for all agents, $\omega_{p_i}(\theta) > 0$ for all θ, the upper-bound is simply given by $\int(1-F(\theta))(1-\tau(\theta))x(\theta)\omega_{p_i}(\theta)dF(\theta)$, which corresponds to optimal price gap of a government with Rawlsian preferences ($\Lambda(\theta) = 1$ for all θ).

Starting from Proposition 2, the same arguments can be used to provide bounds on the optimal taxes on new technology firms, when only those taxes are available.

4 An Example with Robots

To provide further intuition about the forces that shape optimal taxes, we turn to a special case of the economic environment presented in Section 2.

[10] Since we can always change variable and express the above integrals as a function of earnings, this is equivalent to having access to data on the distribution of earnings.

4.1 Economic Environment

We consider an economy with a unique final good. Agents have quasi-linear preferences

$$U = c - h(n). \tag{14}$$

Old technology firms produce the final good using robots and labor,

$$y = \int y(r(\theta), n(\theta); \theta) dF(\theta), \tag{15}$$

where $y(r(\theta), n(\theta); \theta)$ denotes the output of agents of type θ. We assume that $y(\cdot, \cdot; \theta)$ is homogeneous of degree one for all θ. Like in the example of Section 2.2, new technology firms produce robots using the final good,

$$r^* = \phi y^*, \tag{16}$$

where ϕ measures the productivity of robot makers.[11]

We let p_r and p_r^*denote the price of robots faced by old and technology firms and use the final good as our numeraire. Since robots are only demanded by firms, but not agents, these are the only relevant good prices in this economy. Profit maximization by new technology firms implies

$$p_r^* = 1/\phi,$$

whereas profit maximization by old technology firms implies

$$1 = z(w(\theta), p_r; \theta),$$

where $z(p_r, w(\theta); \theta) \equiv \min_{r,n}\{p_r r + w(\theta)n | y(r, n; \theta) \geq 1\}$ denotes the unit costs of firms using agents of type θ. Note that the wage $w(\theta)$ of any agent θ only depends on the price of robots, faced by old technology firms, not the labor supply decisions of the agent. Here, robots directly affect inequality by affecting relative marginal products of labor, but not indirectly through further changes in relative labor supply.[12]

In line with the notation of the previous sections, we let $w(p_r; \theta)$ denote the equilibrium wage of agent θ as a function of the price of robots, that is the unique solution

[11]For notational convenience, we depart from the convention implicitly used in Sections 2 and 3 where inputs were treated as negative numbers. Here, the quantity of robots, $r(\theta)$ used by old technology firms with a agent of type θ are positive numbers. Likewise, the quantity of final goods, y^*, used to produce robots by new technology firms is also a positive number.

[12]This second effect is the focus of the analysis of optimal income taxes in Stiglitz (1982).

to $1 = z(w(\theta), p_r; \theta)$. Similarly, we let $r(p_r, n(\theta); \theta)$ denote the equilibrium amount of robots used by agents of type θ, that is the solution to $p_r = dy(r(\theta), n(\theta); \theta)/dr$; we let $r(p_r, \{n(\theta)\}) \equiv \int r(p_r, n(\theta); \theta) dF(\theta)$ denote the aggregate demand for robots; and we let $y(p_r, \{n(\theta)\}) \equiv \int y(r(p_r, n(\theta); \theta)), n(\theta); \theta) dF(\theta)$ denote gross output of the final good.

Under the previous assumptions, the planner's problem (8) simplifies into

$$\max_{\{U(\theta)\}, \{n(\theta)\}, p_r} \int U(\theta) d\Lambda(\theta)$$

subject to

$$U'(\theta) = h'(n(\theta)) n(\theta) \omega(p_r; \theta),$$

$$c(\{n(\theta)\}, \{U(\theta)\}) \leq y(p_r, \{n(\theta)\}) - \frac{1}{\phi} r(p_r, \{n(\theta)\}),$$

where $c(\{n(\theta)\}, \{U(\theta)\}) = \int (U(\theta) + h(n(\theta))) dF(\theta)$ denotes the total consumption of the final good, which must be weakly less than gross output minus investment.

4.2 The Optimal Tax on Robots

To characterize the optimal tax on robots, we again start from the first-order condition with respect to p_r. In this environment, it simplifies into

$$\gamma(p_r - p_r^*) r_{p_r}(p_r, \{n(\theta)\}) = \int \mu(\theta) h'(n(\theta)) n(\theta) \omega_{p_r}(p_r; \theta) d\theta. \tag{17}$$

The first-order condition with respect to $n(\theta)$ also takes a simple form. As demonstrated in Appendix A.2, it can be expressed as

$$\gamma[w(\theta)\tau(\theta) + (p_r - p_r^*) r_{n(\theta)}] f(\theta) = \mu(\theta) h'(n(\theta)) [\frac{\epsilon(\theta) + 1}{\epsilon(\theta)}] \omega(p_r; \theta), \tag{18}$$

where $\epsilon(\theta) \equiv \frac{d \ln(n(\theta))}{d \ln h(n(\theta)))} \geq 0$ denotes the Frisch elasticity. Using the previous expression to substitute for $\mu(\theta) h'(n(\theta))/\gamma$ in equation (17) and noting that $\partial \ln r(p_r, n(\theta); \theta)/\partial \ln n(\theta) =$

1,[13] we obtain

$$\frac{p_r - p_r^*}{p_r} = \frac{\int \frac{\epsilon(\theta)}{\epsilon(\theta)+1} \cdot \eta(\theta) \cdot \tau(\theta) \cdot \frac{1-s_r(\theta)}{s_r(\theta)} \cdot g_r(\theta) dF(\theta)}{\rho - \int \frac{\epsilon(\theta)}{\epsilon(\theta)+1} \cdot \eta(\theta) \cdot g_r(\theta) dF(\theta)}, \tag{19}$$

where $\rho \equiv \partial \ln r(p_r, \{n(\theta)\}) / \partial \ln p_r \leq 0$ is the elasticity of the demand for robots; $\eta(\theta) \equiv \partial \ln \omega(p_r; \theta) / \partial \ln p_r$ is the elasticity of relative wages with respect to the price of robots; $s_r(\theta) \equiv p_r r(p_r, n(\theta); \theta) / (p_r r(p_r, n(\theta); \theta) + w(\theta) n(\theta))$ is the share of robots in the costs of firms using agents of type θ; and $g_r(\theta) \equiv r(p_r, n(\theta); \theta) / r(p_r, \{n(\theta)\})$ is the fraction of robots employed with agents of type θ.

Since agents do not consume robots, the previous price gap can be implemented equivalently with a negative tax on old technology firms, a positive tax on new technology firms, or a combination of both. For expositional purposes, we focus in the rest of this section on the case where the tax on old technology firms has been set to zero and refer to $t_r^* = p_r / p_r^* - 1$ as the tax on robots. Given this normalization, equation (19) leads to the following proposition.

Proposition 3. *If equations (14)-(16) hold, the optimal tax on robots satisfies*

$$\frac{t_r^*}{1 + t_r^*} = \frac{\int \frac{\epsilon(\theta)}{\epsilon(\theta)+1} \cdot \eta(\theta) \cdot \tau(\theta) \cdot \frac{1-s_r(\theta)}{s_r(\theta)} \cdot g_r(\theta) dF(\theta)}{\rho(p_r) - \int \frac{\epsilon(\theta)}{\epsilon(\theta)+1} \cdot \eta(\theta) \cdot g_r(\theta) dF(\theta)}.$$

From a qualitative standpoint, the sign of the tax on robots reflects the same considerations as our previous propositions. If cheaper robots tend to be associated with more inequality, $\eta(\theta) < 0$, then a government trying to redistribute income from high- to low-skilled workers has an incentive to set the price of robots above its laissez-faire value, $p_r > 1/\phi$, which corresponds to $t_r^* > 0$. From a quantitative standpoint, the attractive feature from Proposition 3 comes from the fact that it provides a formula that only involves marginal tax rates, $\{\tau(\theta)\}$, employment and cost shares, $\{g_r(\theta), s_r(\theta)\}$, and elasticities, $\{\epsilon(\theta), \eta(\theta), \rho\}$. In principle, all can be either directly observed or estimated.

One might have thought that in order to assess the magnitude of the optimal tax on robots, one would necessarily need to make assumptions about the distribution of the Pareto weights, Λ. Proposition 3 shows that this is not the case. Intuitively, the underlying preferences of the government for the utility of different groups of agents get revealed by

[13] Recall that $r(p_r, n(\theta); \theta)$ is implicitly defined as the solution to $p_r = dy(r(\theta), n(\theta); \theta)/dr$. Since $y(\cdot, \cdot; \theta)$ is homogeneous of degree one, this is equivalent to $p_r = dy(r(\theta)/n(\theta), 1; \theta)/dr$. Differentiating, we therefore get $\partial \ln r(p_r, n(\theta); \theta) / \partial \ln n(\theta) = 1$.

the marginal tax rates, $\{\tau(\theta)\}$, that they face.[14]

4.3 Comparative Statics

Our final set of results explores the relationship between the efficiency of the new technology, ϕ, and the magnitude of optimal taxes on old and new technology firms. As progress in Artificial Intelligence makes for cheaper and better robots, should we tax them more?

To maintain the analysis tractable, we take a first pass at this question in the context of a parametric example with Rawlsian preferences, constant Frisch elasticities, and Cobb-Douglas production functions. Formally, we assume that the distribution of Pareto weights is such that $\Lambda(\theta) = 1$ for all θ; the distribution of types is uniformly distributed between 0 and 1; preferences are such that

$$h(n) = \frac{n^{1+1/\epsilon}}{1+1/\epsilon}, \tag{20}$$

and technology is such that

$$y(r, n; \theta) = \exp(\alpha(\theta)) \cdot (\frac{r}{\beta(\theta)})^{\beta(\theta)} (\frac{n}{1-\beta(\theta)})^{1-\beta(\theta)}, \tag{21}$$

with $\alpha(\theta) \equiv \frac{\alpha \ln(1-\theta)}{\beta \ln(1-\theta)-1}$, $\beta(\theta) \equiv \frac{\beta \ln(1-\theta)}{\beta \ln(1-\theta)-1}$, and $\alpha, \beta > 0$.

In this case, the zero-profit condition of old technology firms leads to

$$w(p_r; \theta) = (1-\theta)^{-1/\gamma(p_r)},$$

with $\gamma(p_r) \equiv 1/(\alpha - \beta \ln p_r)$. Under the restriction that $\gamma(p_r) > 0$, which we maintain throughout, wages are increasing in θ and Pareto distributed with shape parameter equal to $\gamma(p_r)$ and lower bound equal to 1. By construction, more skilled workers tend to use robots relatively more, $\beta(\theta)$ is increasing in θ. So a decrease in the price of robots tends to increase their wages relatively more and increase inequality, as reflected by the fact that $\gamma'(p_r) = \frac{1}{p_r(\alpha - \beta \ln p_r)^2} > 0$.

For comparative static purposes, a limitation of Propositions 1 through 3 is that they all involve the optimal marginal tax rates. These are themselves endogenous object that will respond to productivity shocks. In Appendix A.3, we first derive a tax formula that substitutes for the optimal marginal tax rates by combing the first-order conditions with respect to $U(\theta)$ and $n(\theta)$. As described in the next proposition, this leads to an optimal

[14] This is the idea behind Werning's (2007) test of whether an income tax schedule is Pareto optimal. Namely, it is if the inferred Pareto weights are all positive.

tax on robots that only depends on a few sufficient sufficient statistics.

Proposition 4. *If equations (14)-(21) hold, the optimal tax on robots satisfies*

$$\frac{t_r^*}{1+t_r^*} = \frac{\Phi}{\rho - \Phi}\frac{1-s_r}{s_r},$$

with $\Phi \equiv -[\epsilon\beta\gamma(p_r)]/[(\epsilon+1)+\epsilon\gamma(p_r)]$ *and* s_r *the aggregate robot share.*

Proposition 4 points towards three critical considerations for assessing whether productivity improvements should be associated with higher or lower taxes on robots. First, is the demand for robots becoming more or less elastic? This is the effect captured by ρ. Second, is the aggregate robot share s_r increasing or decreasing? And third, is inequality becoming more or less responsive to changes in the price of robots? This is the effect captured by Φ.

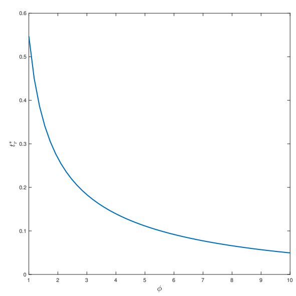
After expressing the three previous statistics as functions of t_r^* and ϕ, we can apply the Implicit Function Theorem to determine the monotonicity of the tax on robots, as we do in Appendix A.3. This leads to our next proposition.

Proposition 5. *If equations (14)-(21) hold, the optimal tax on robots,* t_r^**, decreases with the productivity of robot makers,* ϕ*.*

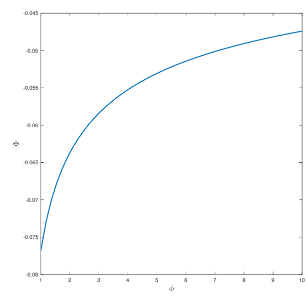
In this parametric example, we have $\partial\Phi/\partial\phi > 0$, $\partial s_r/\partial\phi > 0$, and $\partial\rho/\partial\phi < 0$. The first inequality implies that although cheaper robots always increase inequality, they do so less and less as productivity increases. The second inequality states that the share of robots is increasing. Although the shares of robots employed by agents of any given type remain constant, cheaper robots lead to an increase in the labor supply of high-skill agents. Since they use robots more intensively and their demand is more elastic, this compositional effect increases the aggregate robot share, s_r, and the robot elasticity, ρ. All three effects push towards a lower tax on robots, as described in Figure 1 . As this example illustrates, cheaper robots may lead to a higher share of robots in the economy, more inequality, but a lower optimal tax on robots.

For readers more interested in globalization than automation, we note that robots in this example may be reinterpreted as machines imported from China, with ϕ capturing the relative price of those machines in terms of the final good.[15] Another direct implication of our example therefore is that more globalization and more inequality, in spite of the government having extreme distributional concerns and globalization causing inequality, may be optimally met with less trade protection.

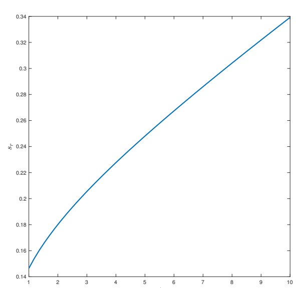
[15] Burstein et al. (2013) develop a model of trade in intermediate goods along those lines.



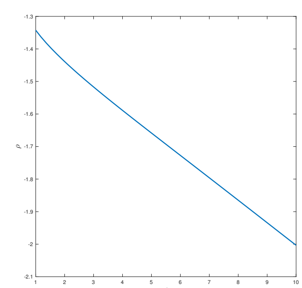
(a) Optimal tax on robots, t_r^*



(b) Responsiveness of Inequality, Φ



(c) Aggregate robot share, s_r



(d) Robot elasticity, ρ

Figure 1: Comparative statics with respect to ϕ.

5 Evaluating New Technologies

In the previous sections, we have focused on the structure of optimal taxes. We now turn to the evaluation of the welfare impact of a change in the productivity of new technology firms under the assumption that constrained, but optimal policies are in place. We do so in the general environment of Section 2 and no longer restrict ourselves to non-linear income taxation as in Section 3 and 4. As will soon be clear, many of the assumptions of Section 2 can themselves be relaxed further.

5.1 An Envelope Result

Consider first a small productivity shock from ϕ to $\phi + d\phi$. Mathematically, evaluating the welfare impact of such a shock is a straightforward matter. Let $W(\phi)$ denote the value function associated with our general planning problem,

$$W(\phi) = \max_{\{U(\theta)\},\{n(\theta)\},R\in\mathcal{R},\{p_i,q_i\}\in\mathcal{P}} \int U(\theta)d\Lambda(\theta)$$

subject to

$$n(\theta), U(\theta) \in \text{argmax}_{n,U}\{U|e(\{q_i\}, C(n,U)) = R(w(\{p_i\},\{n(\theta)\};\theta)n;\theta)\},$$

$$G^*(\{c_j(\{q_i\},\{n(\theta)\},\{U(\theta)\}) - y_j(\{p_i\},\{n(\theta)\})\};\phi) \leq 0.$$

The Envelope Theorem immediately implies

$$\frac{dW}{d\phi} = \gamma\frac{\partial G^*}{\partial\phi}.$$

This leads to our next proposition.

Proposition 6. *Technological change increases social welfare, $dW/d\phi \geq 0$, if and only if it expands the production possibility of new technology firms, that is, if and only if $\frac{\partial G^*}{\partial\phi} \leq 0$.*

Proposition 6 can be thought of as a generalization of Hulten's (1978) Theorem—which is a direct implication of the Envelope Theorem in first-best economies—to distorted economies. Because of restrictions on the set of feasible income tax schedules and prices, as captured by $\mathcal{R}$ and $\mathcal{P}$, our economy may be distorted. Marginal rates of substitutions may not be equalized across agents and marginal rates of transformation may not be equalized between and old technology firms. Yet, the welfare impact of technological progress can be measured in the exact same way as in first-best environments, with unrestricted tax instruments, where social welfare is given by

$$W_{1^{st}best}(\phi) = \max_{\{U(\theta)\},\{n(\theta)\},\{p_i,q_i\}} \int U(\theta) d\Lambda(\theta)$$

subject to

$$G^*(\{c_j(\{q_i\},\{n(\theta)\},\{U(\theta)\}) - y_j(\{p_i\},\{n(\theta)\})\};\phi) \leq 0.$$

Accordingly, technological progress among new technology firms, either robot producers or international traders, cannot be socially harmful.

5.2 Why Can't Technological Progress Hurt Us?

It is well-known that, in the presence of distortions, the opposite may happen. This is what Edgeworth (1884) refers to as "Economic Damnification" and what Bhagwati (1958) refers to as "Immesirizing Growth." As discussed by Bhagwati (1971), such situations arise whenever the direct benefits of technological progress are outweighed by the costs of aggravating some underlying distortion. In Bhagwati (1958) and Johnson (1967), this occurs because of sector-specific growth, when an optimal tariff is not in place. In Hagen (1958), this happens because of opening up trade, when wages are not equalized across sectors. So how did we escape damnification?

The critical assumption here is not that there are no distortions or, equivalently, that our planner has enough tax instruments to target any underlying distortion. We can, in fact, be quite far from the assumptions of the First and Second Welfare Theorems. Proposition 6 instead builds on the assumption that, in spite of distortions and restrictions on the set of available instruments, our planner still has enough tax instruments, namely $\{t_i^*\}$, to control fully the behavior of new technology firms, which is where technological change is occurring.

Take the example of international trade: $G^*(\{y_i^*\};\phi) = \sum \bar{p}_i(\phi) y_i^*$. In this case, Proposition 6 states that up to a first-order approximation, a country gains from a terms-of-trade shock if and only if it raises the value of its net exports, evaluated at the initial quantities, $\sum \frac{d\bar{p}_i(\phi)}{d\phi} y_i^*$.[16] This is the exact same expression as in the laissez-faire equilibrium of a perfectly competitive model of international trade. Provided that the government optimally chooses trade policy, our analysis implies that whether the China Shock is good or bad for the U.S. economy can be evaluated from its terms-of-trade effect alone.

Intuitively, if world prices , $\{\bar{p}_i(\phi)\}$, change, the U.S. government always has the op-

[16] The previous observation does not depend on whether the country is a small open economy or not. $\{\bar{p}_i(\phi)\}$ could themselves be functions of $\{y_i^*\}$. The same result would hold.

tion to maintain prices in the United States, $\{q_i, p_i\}$, unchanged by raising trade taxes by $\{\frac{d\bar{p}_i(\phi)}{d\phi}\}$. Starting from a constrained optimum, the only first-order effect of such a policy change would be to raise tax revenues by $\sum \frac{d\bar{p}_i(\phi)}{d\phi} y_i^*$, regardless of whether or not the U.S. economy is first-best. Raising trade taxes by $\{\frac{d\bar{p}_i(\phi)}{d\phi}\}$, of course, may not be the optimal response, but the possibility of such a response is sufficient to evaluate the welfare impact of a terms-of-trade shock at the margin.

It should also be clear that the previous envelope is much more general than what the assumptions imposed in Section 2 may suggest. Mathematically, a weaker sufficient condition for Proposition 6 to hold is the existence of a set of feasible allocation, $\mathcal{Z}$, independent of ϕ, such that the planner's problem can be expressed as

$$\max_{\{U(\theta), n(\theta), c_j(\theta), y_j\} \in \mathcal{Z}} \int U(\theta) d\Lambda(\theta)$$

subject to

$$G^*(\{\int c_j(\theta) dF(\theta) - y_j\}; \phi) \leq 0.$$

This general formulation allows for arbitrary preferences and technology across agents and old technology firms. It also allows for production and consumption externalities as well as various market imperfections. In particular, there may be price and wage rigidities leading to labor market distortions. The first-order welfare effect of a productivity shock remains given by $\gamma \frac{\partial G^*}{\partial \phi}$.

5.3 Measuring The Welfare Gains from Trade and Robots

Our next goal is to show how to go from the previous envelope result to the welfare analysis of potentially large technological shocks such as those brought about by globalization and automation. The basic idea is to follow, in a general equilibrium environment with distortions, the same steps used to compute equivalent and compensating variations associated with exogenous price changes in standard consumer theory.

Consider the following generalized version of our planning problem

$$W(\phi, D) = \max_{\{U(\theta)\}, \{n(\theta)\}, R \in \mathcal{R}, \{p_i, q_i\} \in \mathcal{P}} \int U(\theta) d\Lambda(\theta)$$

subject to

$$n(\theta), U(\theta) \in \operatorname{argmax}_{n,U} \{U | e(\{q_i\}, C(n, U)) = R(w(\{p_i\}, \{n(\theta)\}; \theta) n; \theta)\},$$

$$G^*(\{c_j(\{q_i\},\{n(\theta)\},\{U(\theta)\}) - y_j(\{p_i\},\{n(\theta)\})\};\phi) \le D.$$

The parameters ϕ and D play the same role here as prices and income in the utility maximization problem of a single consumer. In our international trade example, which we come back to below, D simply corresponds to a trade deficit, that is a transfer from the rest of the world. The welfare impact of a productivity shock from ϕ_0 to ϕ_1 can then be computed either as the transfer, $EV(\phi_0,\phi_1)$, that would be equivalent to the shock, $W(\phi_0, EV(\phi_0,\phi_1)) = W(\phi_1, 0)$, or as the transfer, $CV(\phi_0,\phi_1)$, required to compensate for the shock, $W(\phi_1, -CV(\phi_0,\phi_1)) = W(\phi_0, 0)$.

Let $\{y_i^*(\phi, D)\}$ denote the vector of output by new technology firms associated with the solution to the generalized planner's problem. Our envelope result states that for infinitesimal changes, we must have

$$EV(\phi,\phi + d\phi) = CV(\phi,\phi + d\phi) = \frac{\partial G^*(\{y_i^*(\phi,0)\};\phi)}{\partial\phi}.$$

This is the counterpart to Shephard's lemma in standard consumer theory. And, like in standard consumer theory, the previous envelope condition can be integrated to compute the welfare impact of large technological changes.

Proposition 7. *The equivalent variation, $EV(\phi_0,\phi_1)$, associated with a productivity shock from ϕ_0 to ϕ_1 corresponds to the unique solution to the differential equation*

$$\frac{dEV(\phi,\phi_1)}{d\phi} = \frac{\partial G^*(\{y_j^*(\phi, EV(\phi,\phi_1))\};\phi)}{\partial\phi}, \text{ with initial condition } EV(\phi_1,\phi_1) = 0,$$

evaluated at $\phi = \phi_0$. Likewise, the compensating variation, $CV(\phi_0,\phi_1)$, corresponds to the unique solution to

$$\frac{dCV(\phi_0,\phi)}{d\phi} = \frac{\partial G^*(\{y_j^*(\phi, -CV(\phi_0,\phi))\};\phi)}{\partial\phi}, \text{ with initial condition } CV(\phi_0,\phi_0) = 0,$$

evaluated at $\phi = \phi_1$.

In the previous expressions, $\{y_i^*(\phi, EV(\phi,\phi_1))\}$ and $\{y_i^*(\phi, -CV(\phi_0,\phi))\}$ are the counterparts to the compensated Hicksian demand functions, evaluated at the final and the initial utility level, respectively. In the same way that knowledge of the Marshallian demand curve is sufficient to compute the welfare impact of price changes, Proposition 7 establishes that knowledge of $\{y_i^*(\phi, D)\}$, i.e., the demand for the goods produced by new technology firms, is sufficient to compute the welfare gains from a productivity shock from ϕ_0 to ϕ_1.

Distortions may, of course, affect $\{y_i^*(\phi, D)\}$ at the solution to our planner's problem. So, the point here is not that distortions do not matter for the welfare consequences of globalization or automation. The point rather is that like in a first-best environment, the demand for goods produced using the new technology, either Chinese imports or robots, fully reveals the welfare gains associated with that technology. Concerns for redistribution and other potential sources of distortions may affect how much we trade or how much we use robots, but not the mapping between quantities demanded, productivity shocks, and welfare.

5.4 Managing Innovation?

To illustrate further the usefulness of our envelope result, we conclude our analysis by turning to the issue of whether governments should also try to manage innovation because of its adverse distributional consequences.

To shed light on this issue in the simplest possible way, suppose that there exists a set of feasible new technologies, Φ, that can be restricted by the government. The profit maximization problem of new technology firms is now given by

$$\{y_i^*, \phi^*\} \in \arg \max_{\{\tilde{y}_i\}, \phi \in \bar{\Phi}} \{\sum p_i^* \tilde{y}_i | G^*(\{\tilde{y}_i\}; \phi) \leq 0\},$$

where $\bar{\Phi} \subset \Phi$ is the set of technologies allowed by the government. The rest of our analysis is unchanged.

In this environment, the general planning problem of Section 5.1 becomes

$$\max_{\phi \in \Phi, \{U(\theta), n(\theta), c_j(\theta), y_j\} \in \mathcal{Z}} \int U(\theta) d\Lambda(\theta)$$

subject to

$$G^*(\{\int c_j(\theta) dF(\theta) - y_j\}; \phi) \leq 0.$$

The optimal technology, ϕ^*, therefore simply satisfies

$$\frac{\partial G^*(\{y_j^*\}; \phi^*)}{\partial \phi} = 0.$$

Provided that taxes of new technology firms, $\{t_j^*\}$, have been set such that they find it optimal to produce $\{y_j^*\}$, conditional on ϕ^*, they will also find it optimal to choose ϕ^* , if allowed to do so. It follows that the government does not need to affect the direction of innovation, in spite of its potential distributional implications.[17]

6 Conclusion

Our paper focuses on two broad sets of issues. The first one is related to the management of new technologies. Among other things, we have asked: should we tax or subsidize firms using new technologies? Should we tax or subsidize firms developing these technologies? And to the extent that taxes should be imposed, what are the observables that can guide the optimal taxation of new technology firms in practice?

In second best environments—where income taxation is available, but specific factor taxes are not—we have shown that there may be a case for taxing new technology firms, if taxes on old technology firms are unavailable, but that distributional concerns do not create a new rationale for taxing or subsidizing (the direction of) innovation. We have derived a number of optimal tax formulas, including a formula for the optimal tax on robots that dispenses with any assumption on the distribution of Pareto weights in the population. This is empirically appealing. We have also used our formulas to conduct comparative statics and illustrate through a parametric example that more robots, or more trade, may go hand in hand with more inequality and lower taxes on robots and trade flows.

The second set of issues is related to the evaluation of new technologies. In spite of tax instruments being limited and the government having concerns for redistribution, we show that productivity shocks can be evaluated using a simple envelope argument, like in first best environments. This stands in sharp contrast with the existing results of a large literature concerned with distortions and welfare. Our results imply, in particular, that the welfare gains from trade or the welfare gains from the introduction of robots can still be computed by integrating below the demand curves for foreign goods and robots, respectively. Distributional concerns and various distortions may affect how much we trade or how much we use robots, but not the welfare implications from changes in the demand for those.

It goes without saying that the extent to which our envelope result is useful in practice depends on whether constrained, but optimal policies are in place. The assumption that the government fully controls the new technology clearly is a strong one. We view it, however, as a useful benchmark, not necessarily stronger than the opposite assump-

[17]More generally, there may be externalities across firms that directly call for subsidizing, or taxing, innovation. In such environments, the implication of our envelope result is that distributional concerns do not a new motive for taxing, or subsidizing, innovation.

tion, implicit in many papers, that governments cannot control the new technology at all. Antras, de Gortari and Itskhoki (2017), Galle et al. (2017), and Waugh and Lyon (2017) are recent welfare analysis of the so-called China shock that fall into this category. Brexit and the current debate about the renegotiation of NAFTA are stark reminders that trade policies are not set in stone.

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A Proofs

A.1 Section 3.3

The Lagrangian associated with the planner's problem (8) is given by

$$\mathcal{L} = \int U(\theta)d\Lambda(\theta) + \int \mu(\theta)\left(U'(\theta) + u_n(C(n(\theta),U(\theta)),n(\theta))n(\theta)\omega(\{p_i\},\{n(\theta)\})\right)d\theta \\ - \gamma G^*(\{c_j(\{q_i\},\{n(\theta)\},\{U(\theta)\}) - y_j(\{p_i\},\{n(\theta)\})\};\phi).$$

Integrating by parts, we get

$$\mathcal{L} = \int U(\theta)d\Lambda(\theta) - \int \mu'(\theta)U(\theta)d\theta + U(\bar{\theta})\mu(\bar{\theta}) - U(\underline{\theta})\mu(\underline{\theta}) \\ + \int \mu(\theta)u_n(C(n(\theta),U(\theta)),n(\theta))n(\theta)\omega(\{p_i\},\{n(\theta)\})d\theta \\ - \gamma G^*(\{c_j(\{q_i\},\{n(\theta)\},\{U(\theta)\}) - y_j(\{p_i\},\{n(\theta)\})\};\phi).$$

Since $U(\bar{\theta})$ and $U(\underline{\theta})$ are free we must have

$$\mu(\underline{\theta}) = \mu(\bar{\theta}) = 0.$$

The first-order condition with respect to $U(\theta)$ leads to

$$\lambda(\theta) - \mu'(\theta) + \mu(\theta)u_{nC}(\theta)C_{U(\theta)}(\theta)n(\theta)\omega(\theta) - \gamma\nabla_{y^*}G^* \cdot c_{U(\theta)}f(\theta) = 0,$$

where λ denotes the density associated with Λ. From condition (5) and the normalization $p_1^* = 1$, we know that $p_i^* = G^*_{y_i}/G^*_{y_1}$ for all i. From the fact that $D_q c \cdot (p^* - q) = 0$ at the optimum, as argued in the main text, and the normalization $q_1 = 1$, we also know that $p_i^* = q_i$ for all i. Using these two observations, we get

$$\mu'(\theta) - \mu(\theta)u_{nC}(\theta)C_{U(\theta)}(\theta)n(\theta)\omega(\theta) = \lambda(\theta) - \gamma\zeta(\theta)f(\theta).$$

where $\zeta(\theta) \equiv G^*_{y_1}\sum q_j \frac{dc_j(\{q_i\},C(n(\theta),U(\theta)))}{dU(\theta)}$ denote the inverse of the marginal utility of income. Let $\tilde{u}(C,x,\theta) \equiv u(C,x/w(\theta))$, $\tilde{u}_\theta \equiv \frac{\partial\tilde{u}}{\partial\theta}$, and $\tilde{u}_{\theta C} \equiv \frac{\partial^2\tilde{u}}{\partial\theta\partial C}$. By definition, we have

$$\tilde{u}_\theta(\theta) = -u_n(\theta)n(\theta)\omega(\theta)$$

and

$$\tilde{u}_{\theta C}(\theta) = -u_{nC}(\theta)n(\theta)\omega(\theta)$$

Using the previous notation and using the fact that $C_{U(\theta)} = 1/u_C(\theta) = 1/\tilde{u}_C(\theta)$, we can rearrange the above first-order condition as

$$\mu'(\theta)\tilde{u}_C(\theta) + \mu(\theta)\tilde{u}_{\theta C}(\theta) = \tilde{u}_C(\theta)(\lambda(\theta) - \gamma\zeta(\theta)f(\theta)). \tag{22}$$

Let $\tilde{\mu}(\theta) \equiv \mu(\theta)\tilde{u}_C(\theta)$. By definition, we also have

$$\tilde{\mu}'(\theta) = \mu'(\theta)\tilde{u}_C(\theta) + \mu(\theta)\tilde{u}'_C(\theta)$$

with

$$\tilde{u}'_C(\theta) = \tilde{u}_{\theta C}(\theta) + \tilde{u}_{CC}(\theta)C'(\theta) + \tilde{u}_{Cx}(\theta)x'(\theta),$$

which can be rearranged as

$$\tilde{u}'_C(\theta) = \tilde{u}_{\theta C}(\theta) + x'(\theta)[\tilde{u}_{CC}(\theta)\frac{dC}{dx} + \tilde{u}_{Cx}(\theta)]. \tag{23}$$

From agent θ's budget constraint (3), we know that

$$\frac{dC}{dx} = \frac{R'(x)}{e_C},$$

and from the first-order condition associated with (2), we know that

$$\frac{R'(x)}{e_C} = -\frac{\tilde{u}_x}{\tilde{u}_C}.$$

Combining the three previous equations, we obtain

$$\tilde{u}'_C(\theta) = \tilde{u}_{\theta C}(\theta) + \tilde{u}_C(\theta)x'(\theta)\rho(\theta), \tag{24}$$

where $\rho(\theta) \equiv \frac{\partial(\tilde{u}_x/\tilde{u}_C)}{\partial C} = \frac{\tilde{u}_{Cx}}{\tilde{u}_C} - \tilde{u}_{CC}\frac{\tilde{u}_x}{\tilde{u}_C^2}$ denotes the partial derivative, with respect to aggregate consumption, of the marginal rate of substitution between earnings and consumption. In turn, equations (22), (23), and (24) imply

$$\tilde{\mu}'(\theta) - \tilde{\mu}(\theta)\frac{dx}{d\theta}\rho(\theta) = \tilde{u}_C[\lambda(\theta) - \gamma\zeta(\theta)f(\theta)].$$

Solving forward and using the fact that $\tilde{\mu}(\bar{\theta}) = 0$, we get

$$\begin{aligned}\tilde{\mu}(\theta) = &-\int_\theta^{\bar{\theta}} \exp[-\int_\theta^z \rho(v)dx(v)]\tilde{u}_C(z)d\Lambda(z) \\ &+ \gamma\int_\theta^{\bar{\theta}} \exp[-\int_\theta^z \rho(v)dx(v)]\tilde{u}_C(z)\zeta(z)dF(z).\end{aligned}$$

Since $\tilde{\mu}(\underline{\theta}) = 0$, we must also have

$$\gamma = \frac{\int_{\underline{\theta}}^{\bar{\theta}} \exp[-\int_{\theta}^{z} \rho(v)dx(v)]\tilde{u}_C(z)d\Lambda(z)}{\int_{\underline{\theta}}^{\bar{\theta}} \exp[-\int_{\theta}^{z} \rho(v)dx(v)]\tilde{u}_C(z)\zeta(z)dF(z)},$$

which implies

$$\begin{aligned}\frac{\tilde{\mu}(\theta)}{\gamma} &= \int_{\theta}^{\bar{\theta}} \exp[-\int_{\theta}^{z} \rho(v)dx(v)]\tilde{u}_C(z)\zeta(z)dF(z) \\ &\quad - \frac{\int_{\theta}^{\bar{\theta}} \exp[-\int_{\theta}^{z} \rho(v)dx(v)]\tilde{u}_C(z)d\Lambda(z)}{\int_{\underline{\theta}}^{\bar{\theta}} \exp[-\int_{\underline{\theta}}^{z} \rho(v)dx(v)]\tilde{u}_C(z)d\Lambda(z)} \cdot \int_{\underline{\theta}}^{\bar{\theta}} \exp[-\int_{\underline{\theta}}^{z} \rho(v)dx(v)]\tilde{u}_C(z)\zeta(z)dF(z).\end{aligned}$$

Substituting into equation (9) and using the fact that $u_n(\theta)/u_C(\theta) = -(1-\tau(\theta))w(\theta)$, we obtain

$$D_p y \cdot (p^* - p) = \int \psi(\theta)(1-\tau(\theta))x(\theta)(\nabla_p \omega(\theta))d\theta,$$

with $\psi(\theta)$ given by equation (12), as argued in the main text.

A.2 Section 4.2

The first-order condition with respect to $n(\theta)$ is given by

$$\begin{aligned}\gamma[y_{n(\theta)}(p_r, \{n(\theta)\}) - \frac{1}{\phi} r_{n(\theta)}(p_r, \{n(\theta)\} - c_{n(\theta)}(\{n(\theta)\}, \{U(\theta)\})] \\ = \mu(\theta)[h''(n(\theta))n(\theta) + h'(n(\theta))]\omega(p_r;\theta).\end{aligned} \tag{25}$$

Since old technology firms choose their labor demand to maximize profits and agents choose their labor supply to maximize utility, we also know that

$$\begin{aligned}y_{n(\theta)}(p_r, \{n(\theta)\}) - p_r r_{n(\theta)}(p_r, \{n(\theta)\} &= w(\theta)f(\theta), \\ c_{n(\theta)}(\{n(\theta)\}, \{U(\theta)\}) &= w(\theta)(1-\tau(\theta))f(\theta).\end{aligned}$$

Thus, using the fact that $p_r^* = 1/\phi$, we can rearrange equation (25) into equation (18).

A.3 Section 4.3

Proposition 4 Under the assumption that upper-level preferences are quasilinear, we have already argued that

$$\frac{\mu(\theta)}{\gamma} = \Lambda(\theta) - F(\theta).$$

Together with equation (18), we therefore get

$$[w(\theta)\tau(\theta) + (p_r - p_r^*)r_{n(\theta)}] = \frac{[\Lambda(\theta) - F(\theta)]}{f(\theta)} h'(n(\theta))[\frac{\epsilon(\theta)+1}{\epsilon(\theta)}]\omega(p_r;\theta).$$

Using again the fact that $\partial \ln r(p_r, n(\theta); \theta)/\partial \ln n(\theta) = 1$ and $h'(n(\theta)) = w(\theta)(1-\tau(\theta))$, from the first-order condition of the agent's utility maximization problem, this leads to

$$\tau(\theta) = \frac{1 - \frac{p_r - p_r^*}{p_r} \cdot \frac{s_r(\theta)}{1-s_r(\theta)} \cdot \frac{\epsilon(\theta)}{\epsilon(\theta)+1} \cdot \frac{f(\theta)}{(\Lambda(\theta)-F(\theta))\omega(p_r;\theta)}}{1 + \frac{\epsilon(\theta)}{\epsilon(\theta)+1} \cdot \frac{f(\theta)}{(\Lambda(\theta)-F(\theta))\omega(p_r;\theta)}}. \tag{26}$$

In the absence of taxes on robots, $\frac{t_r^*}{1+t_r^*} = \frac{p_r - p_r^*}{p_r} = 0$, this reduces to the optimal tax schedule in Diamond (1998), Saez (2001), and Scheuer and Werning (2017).

Combining the previous expression with Proposition 3, we obtain

$$\frac{t_r^*}{1+t_r^*} = \frac{\int \Phi(\theta) \cdot \frac{1-s_r(\theta)}{s_r(\theta)} \cdot g_r(\theta) dF(\theta)}{\rho - \int \Phi(\theta) \cdot g_r(\theta) dF(\theta)} \tag{27}$$

with $\Phi(\theta) \equiv \frac{\epsilon(\theta)\eta(\theta)(\Lambda(\theta)-F(\theta))\omega(p_r;\theta)}{(\epsilon(\theta)+1)(\Lambda(\theta)-F(\theta))\omega(p_r;\theta)+\epsilon(\theta)f(\theta)}$. In the parametric example of Section 4.3, we have assumed

$$\epsilon(\theta) = \epsilon \text{ for all } \theta, \tag{28}$$

$$\Lambda(\theta) = 1 \text{ for all } \theta, \tag{29}$$

$$f(\theta) = 1 \text{ for all } \theta, \tag{30}$$

$$F(\theta) = \theta \text{ for all } \theta. \tag{31}$$

We therefore immediately get

$$\Phi(\theta) = \frac{\epsilon\eta(\theta)(1-\theta)\omega(p_r;\theta)}{(\epsilon+1)(1-\theta)\omega(p_r;\theta)+\epsilon}. \tag{32}$$

In Section 4.3, we have also established that

$$w(p_r;\theta) = (1-\theta)^{-1/\gamma(p_r)}.$$

This implies

$$\omega(p_r;\theta) = \frac{d \ln w(\theta; p_R)}{d\theta} = \frac{1}{\gamma(p_r)} \cdot \frac{1}{1-\theta},$$

$$\eta(\theta) = \frac{d \ln \omega(p_r;\theta)}{d \ln p_r} = -\frac{d \ln \gamma(p_r)}{d \ln p_r} = -\beta\gamma(p_r),$$

where the last equality uses the definition of $\gamma(p_r) \equiv 1/(\alpha - \beta \ln p_r)$. Substituting into equation (32), we therefore get

$$\Phi(\theta) = -\frac{\epsilon\beta\gamma(p_r)}{(\epsilon+1)+\epsilon\gamma(p_r)} \equiv \Phi. \tag{33}$$

Combining equations (27) and (33), we obtain

$$\frac{t_r^*}{1+t_r^*} = \frac{\Phi}{\rho-\Phi}\frac{1-s_r}{s_r}, \tag{34}$$

where $s_r \equiv \frac{\int p_r r(p_r,n(\theta);\theta)dF(\theta)}{\int [p_r r(p_r,n(\theta);\theta)+w(\theta)n(\theta)]dF(\phi)}$ denotes the aggregate share of robots.

Proposition 5 We first demonstrate that Φ, s_r, and ρ can be expressed as functions of t_r^* and ϕ.

Using the fact that $p_r = (1+t_r^*)/\phi$, we can immediately rearrange equation (33) as

$$\Phi = -\frac{\epsilon\beta\gamma((1+t_r^*)/\phi)}{(\epsilon+1)+\epsilon\gamma((1+t_r^*)/\phi)} \equiv \Phi(t_r^*,\phi). \tag{35}$$

To express s_r and ρ as a function of t_r^* and ϕ, we further need to solve for the optimal labor supply of each agent, $n(\theta)$, which itself depends on the marginal income tax rates, $\tau(\theta)$. Together with equations (28)-(31), equation (26) implies

$$\tau(\theta) = \frac{\epsilon+1-\frac{t_r^*}{1+t_r^*}\frac{s_r(\theta)}{1-s_r(\theta)}\gamma(p_r)}{\epsilon+1+\epsilon\gamma(p_r)}.$$

From the first-order condition of the old technology firms, we know that

$$\frac{s_r(\theta)}{1-s_r(\theta)} = -\beta(\theta)\ln(1-\theta), \tag{36}$$

which leads to

$$\tau(\theta) = \frac{\epsilon+1+\frac{t_r^*}{1+t_r^*}\beta\gamma(p_r)\ln(1-\theta)}{\epsilon+1+\epsilon\gamma(p_r)}. \tag{37}$$

The optimal labor supply is given by the agent's first-order condition

$$n(\theta) = ((1-\tau(\theta))w(\theta))^{\epsilon}. \tag{38}$$

Combining equations (37) and (38) with the fact that $w(p_r;\theta) = (1-\theta)^{-1/\gamma(p_r)}$, we get

$$n(\theta) = (\frac{\gamma(p_r)}{\epsilon+1+\epsilon\gamma(p_r)})^{\epsilon}(\epsilon - \frac{t_r^*}{1+t_r^*}\beta\ln(1-\theta))^{\epsilon}(1-\theta)^{-\epsilon/\gamma(p_r)},$$

and in turn,

$$\int w(\theta)n(\theta)d\theta = (\frac{\gamma(p_r)}{\epsilon+1+\epsilon\gamma(p_r)})^{\epsilon}\int(\epsilon - \frac{t_r^*}{1+t_r^*}\beta\ln(1-\theta))^{\epsilon}\theta^{-\frac{1+\epsilon}{\gamma(p_r)}}d\theta$$

Using equation (36), we further get

$$p_r r(p_r, n(\theta);\theta) = -\beta\ln(1-\theta)(\frac{\gamma(p_r)}{\epsilon+1+\epsilon\gamma(p_r)})^{\epsilon}(\epsilon - \frac{t_r^*}{1+t_r^*}\beta\ln(1-\theta))^{\epsilon}(1-\theta)^{-\frac{1+\epsilon}{\gamma(p_r)}}, \tag{39}$$

and in turn,

$$p_r r(p_r, \{n(\theta)\}) = -\beta(\frac{\gamma(p_r)}{\epsilon+1+\epsilon\gamma(p_r)})^{\epsilon}\int\ln(1-\theta)(\epsilon - \frac{t_r^*}{1+t_r^*}\beta\ln(1-\theta))^{\epsilon}(1-\theta)^{-\frac{1+\epsilon}{\gamma(p_r)}}d\theta. \tag{40}$$

The aggregate share of robots is therefore given by

$$s_r = \frac{\int\beta\ln(1-\theta)(\epsilon - \frac{t_r^*}{1+t_r^*}\beta\ln(1-\theta))^{\epsilon}(1-\theta)^{-\frac{1+\epsilon}{\gamma((1+t_r^*)/\phi)}}d\theta}{\int(\beta\ln(1-\theta)-1)(\epsilon - \frac{t_r^*}{1+t_r^*}\beta\ln(1-\theta))^{\epsilon}(1-\theta)^{-\frac{1+\epsilon}{\gamma((1+t_r^*)/\phi)}}d\theta} \equiv s_r(t_r^*, \phi), \tag{41}$$

where we have again used $p_r = (1+t_r^*)/\phi$. The robot elasticity ρ can be computed in a similar manner. From equation (36) and the fact that $w(p_r;\theta) = (1-\theta)^{-1/\gamma(p_r)}$, we get

$$p_r r(p_r, n(\theta);\theta) = -\beta\ln(1-\theta)n(\theta)(1-\theta)^{-1/\gamma(p_r)}.$$

Using the previous expression with the definition of $\rho \equiv \frac{\partial\ln r(p_r,\{n(\theta)\})}{\partial\ln p_r}$, we get

$$\rho = \int\frac{r(p_r, n(\theta);\theta)}{r(p_r, \{n(\theta)\})}\frac{d\ln w(p_r;\theta)}{d\ln p_r}d\theta - 1.$$

Combining the previous expressions with equations (39), (40), and using the fact that $\frac{d\ln w(p_r;\theta)}{d\ln p_r} = -\beta\ln(1-\theta)$ and $p_r = (1+t_r^*)/\phi$, we get

$$\rho = \frac{\int(\beta\ln(1-\theta)-1)\ln(1-\theta)(\epsilon - \frac{t_r^*}{1+t_r^*}\beta\ln(1-\theta))^{\epsilon}(1-\theta)^{-\frac{1+\epsilon}{\gamma((1+t_r^*)/\phi)}}d\theta}{\int\ln(1-\theta)(\epsilon - \frac{t_r^*}{1+t_r^*}\beta\ln(1-\theta))^{\epsilon}(1-\theta)^{-\frac{1+\epsilon}{\gamma((1+t_r^*)/\phi)}}d\theta} \equiv \rho(t_r^*, \phi). \tag{42}$$

At this point, we have established that the three statistics in equation (34) can be expressed as $\Phi(t_r^*, \phi)$, $\rho(t_r^*, \phi)$, and $s_r(t_r^*, \phi)$. We can therefore rearrange equation (34) as

$$H(t_r^*, \Phi(t_r^*, \phi), \rho(t_r^*, \phi), s_r(t_r^*, \phi)) = 0,$$

with

$$H(t_r^*, \Phi, \rho, s_r) \equiv \frac{\Phi}{\rho - \Phi}\cdot\frac{1-s_r}{s_r} - \frac{t_r^*}{1+t_r^*}.$$

By the Implicit Function Theorem, we have

$$\frac{dt_r^*}{d\phi} = -\frac{dH/d\phi}{dH/dt_r^*}. \tag{43}$$

Since the tax on robots is chosen to maximize welfare, the second derivative of the government's value function, expressed as a function of t_r^* only, must be negative. Noting that H corresponds to its first derivative—which is equal to zero at the optimal tax—we therefore obtain

$$dH/dt_r^* < 0. \tag{44}$$

Since $\gamma(\cdot)$ is a strictly increasing function, equation (35) implies

$$\frac{\partial \Phi(t_r^*, \phi)}{\partial \phi} > 0. \tag{45}$$

To establish the monotonicity of s_r and ρ with respect to ϕ, it is convenient to introduce the following function:

$$d(t_r^*, \phi, \zeta; \theta) = (\epsilon - \beta \frac{t_r^*}{1+t_r^*} \ln(1-\theta))^{\epsilon} (1-\theta)^{-\frac{1+\epsilon}{\gamma((1+t_r^*)/\phi)}} (\ln(1-\theta))^{-\zeta}.$$

By construction, d is log-supermodular in (ϕ, ζ, θ). Since log-supermodularity is preserved by integration, the following function,

$$D(\phi, \zeta) = \int d(t_r^*, \phi, \zeta; \theta) d\theta,$$

is also log-supermodular. It follows that

$$\frac{D(\phi, \zeta = 0)}{D(\phi, \zeta = -1)} = \frac{\int (\epsilon - \beta \frac{t_r^*}{1+t_r^*} \ln(1-\theta))^{\epsilon} (1-\theta)^{-\frac{1+\epsilon}{\gamma((1+t_r^*)/\phi)}} d\theta}{\int (\ln(1-\theta))(\epsilon - \beta \frac{t_r^*}{1+t_r^*} \ln(1-\theta))^{\epsilon} (1-\theta)^{-\frac{1+\epsilon}{\gamma((1+t_r^*)/\phi)}} d\theta} \text{ is increasing in } \phi,$$

$$\frac{D(\phi, \zeta = -2)}{D(\phi, \zeta = -1)} = \frac{\int (\ln(1-\theta))^2 (\epsilon - \beta \frac{t_r^*}{1+t_r^*} \ln(1-\theta))^{\epsilon} (1-\theta)^{-\frac{1+\epsilon}{\gamma((1+t_r^*)/\phi)}} d\theta}{\int (\ln(1-\theta))(\epsilon - \beta \frac{t_r^*}{1+t_r^*} \ln(1-\theta))^{\epsilon} (1-\theta)^{-\frac{1+\epsilon}{\gamma((1+t_r^*)/\phi)}} d\theta} \text{ is decreasing in } \phi.$$

Noting that

$$s_r = \frac{1}{1 - \frac{1}{\beta} \frac{D(\phi, \zeta=0)}{D(\phi, \zeta=-1)}},$$

$$\rho = \beta \frac{D(p_R, \zeta = -2)}{D(p_R, \zeta = -1)} - 1,$$

we obtain that

$$\frac{\partial s_r(t_r^*, \phi)}{\partial \phi} > 0, \tag{46}$$

$$\frac{\partial \rho(t_r^*, \phi)}{\partial \phi} < 0. \tag{47}$$

Since $\frac{\partial H}{\partial \Phi} < 0$, $\frac{\partial H}{\partial s_r} < 0$, and $\frac{\partial H}{\partial \rho} > 0$, inequalities (45)-(47) imply

$$\frac{dH}{d\phi} = \frac{\partial H}{\partial \Phi}\frac{\partial \Phi}{\partial \phi} + \frac{\partial H}{\partial s_r}\frac{\partial s_r}{\partial \phi} + \frac{\partial H}{\partial \rho}\frac{\partial \rho}{\partial \phi} < 0.$$

Combining this observation with equation (43) and (44), we conclude that $dt_r^*/d\phi > 0$.