Gravity waves

The solar radiation on average is stronger in the tropics than at high latitudes. With a radiative-convective equilibrium calculation, we find that the atmosphere is warmer in the tropics and colder at higher latitudes. Can the atmosphere remain at rest? Within a column of air, being at rest requires hydrostatic equilibrium. With this, we deduce that there must be horizontal pressure gradient somewhere, which will drive motion. What happens next?

We shall start by considering the adjustment under gravity in a nonrotating system. The concept is illustrated more easily with water. The focus is on getting a conceptual picture of how gravity waves propagate, not the mathematical details. We shall consider the adjustment of a homogeneous fluid of constant depth H that initially has a small displacement of its free surface η. This problem was solved by Laplace (1893). We shall only consider a simplified version of that by assuming the wavelength of the wave is much longer than the depth of the fluid. This is called the long wave approximation or the shallow water approximation. With this, the fluid can be considered approximately in hydrostatic equilibrium. For a homogeneous fluid (constant density), this implies at all heights

$$p' = \rho g \eta$$

For infinitesimal perturbations, the horizontal momentum equation is now (neglecting viscosity, nonlinearity, and only consider one horizontal dimension):

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \tag{1.1}$$

Also recognize that we have two boundary conditions. At the bottom, the normal velocity is zero (w=0). At the top (the free surface) the vertical velocity is $D\eta/Dt$. For infinitesimal perturbations, it is the same as $\partial\eta/\partial t$. Now integrate the continuity equation with respect to depth and use the two boundary conditions, we have

$$\frac{\partial \eta}{\partial t} + H\frac{\partial u}{\partial x} = 0 \tag{1.2}$$

Eliminate u from the above two equations, we have

$$\frac{\partial^2 \eta}{\partial t^2} = c^2\left(\frac{\partial^2 \eta}{\partial x^2}\right)$$

$$c^2 = gH$$

This is a standard wave equation. Assume solutions of the form exp(-iωt+ikx), we have

$$\omega^2 = k^2 c^2$$

This is the dispersion relation. Going back to Eqs. (1.1) and (1.2), they can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix}\eta \\ u\end{pmatrix} + \begin{pmatrix}0 & H \\ g & 0\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\eta \\ u\end{pmatrix} = 0$$

The matrix has two eigenvalues +/-c and the eigenvectors are (1 g/c) and (1 -g/c). Given an initial condition, project them onto the eigenvectors and each component will propagate along its line of characteristics.

As an example, let's look at an initial state of rest, i.e. u(x, t=0)=0, with surface displacement η(x,t=0)=G(x). We have project this onto the two eigenvectors so that the initial condition is:

$$\frac{1}{2}G(x)\begin{pmatrix} 1 \\ g/c \end{pmatrix}+\frac{1}{2}G(x)\begin{pmatrix} 1 \\ -g/c \end{pmatrix}$$

Now consider the evolution of the amplitude of the first eigenvector (1 g/c), which we shall call φ(x,t), we have

$$\varphi_t + c\varphi_x = 0$$

Following the line of characteristics dx/dt=c, we have φ(x,t)=const=G(x-ct). Therefore, the solution for φ(x,t) is:

$$\frac{1}{2}G(x-ct)$$

The second eigenvector can be dealt with similarly. The solution then is

$$\eta = \frac{1}{2}\left[G(x-ct)+G(x+ct)\right]$$

$$u = \frac{1}{2}\frac{g}{c}\left[G(x-ct)+G(x-ct)\right]$$

Now consider an example. The initial displacement is confined to a finite region and is symmetric about x=0. In this case, we will find that behind the wave fronts, there is zero displacement and no motion. The energy initially in the region is "radiated" away. This is illustrated with the matlab script water_wave.m. Note that for deep water waves, different waves numbers have different phase speeds. Such waves are called dispersive. A simpler illustration of group velocity and phase speed is in grp_vel_4.m (taken from Holton, 2004, An Introduction to Dynamic Meteorology).

The simple system of shallow water wave can in fact be used to study tides in narrow channels/gulfs such as the Gulf of California. As pointed out by Lagrange (1781), small amplitude shallow water waves are completely analogous to small amplitude sound waves. So the problem of tides in a narrow channel of uniform width and depth is the same as that of a flute. This can give rise to spectacular tides when the forcing frequency is close to the intrinsic frequencies of the channel. It is fortunate that sound waves (like shallow water waves) are nondispersive (or only weakly dispersive). Otherwise, we would have a hard time understanding each other.

Now if we have two immiscible layers of fluid, the bottom one being denser. One example is oil over water. Here we have two interfaces, one is the air-oil interface and the other is the oil-water interface. Mathematically, perturbations on the two interfaces may be viewed as those on the two normal modes of the system, the first represents a mode where two layers move in the same sense, while the second represents a mode where the two layers move in the opposite sense. For the first interface (surface gravity wave), we may neglect the density difference between oil and water and the problem is the same as before. For the second interface (internal gravity wave), the restoring force is proportional to the density difference between oil and water. With some algebra, one can show that it follows the same equation as before except with gravity changed to

$$g' = g(\rho_2 - \rho_1)/\rho_2$$

This is called reduced gravity. Because g'<<g, the internal waves have far slower wave speeds, as can be demonstrated by simple experiments. The existence of such internal gravity waves provides explanation to why in certain coastal localities, ships are not able to maintain their normal speed: additional energy is needed to generate these waves.

Note in the above case, even though the forcing is at the surface, disturbances are generated at the lower interface. If we imagine many layers of fluids on top of each other, the forcing at the surface will generate disturbances at the lower interfaces, manifested as vertical propagation of gravity waves. The limit of this is the continuously stratified case, and the atmosphere and ocean are in reality continuously stratified.

Consider the case of incompressible fluid with a mean state of rest and constant stratification (due to salt content, for example). We have in two dimensions:

$$\begin{aligned}
&\bar{\rho}u_t = -p_x \\
&\bar{\rho}w_t = -p_z - \rho g \\
&u_x + w_z = 0 \\
&\rho_t + w\frac{d\bar{\rho}}{dz} = 0
\end{aligned}$$

Again assume the solution of the form exp(-iωt+ikx+imz), we have the dispersion relation

$$\omega^2 = k^2N^2/\left(k^2 + m^2\right) \tag{1.3}$$

where N is the buoyancy frequency defined before. Thus (vertically propagating) internal gravity waves can have any frequency between zero and N, the latter being the maximum. With this dispersion relation, the phase speed and the group velocity are:

$$\begin{aligned}
c_x &= \frac{\omega}{k} = \pm\frac{N}{\left(k^2 + m^2\right)^{1/2}} \\
c_z &= \frac{\omega}{m} = \pm\frac{kN/m}{\left(k^2 + m^2\right)^{1/2}} \\
c_{gx} &= \frac{\partial\omega}{\partial k} = \pm\frac{Nm^2}{\left(k^2 + m^2\right)^{3/2}} \\
c_{gz} &= \frac{\partial\omega}{\partial m} = \pm\frac{-Nkm}{\left(k^2 + m^2\right)^{3/2}}
\end{aligned} \tag{1.4}$$

They are different so the waves are dispersive. Note that we use the term phase speed rather than phase velocity because it is not a vector. The phase speed in a new direction is not the vector sum of the phase speeds in x and z directions.

For internal gravity waves, in fact the vertical group velocity and the vertical phase speed are of opposite sign.

As group velocity indicates the direction of energy propagation, the direction of energy propagation is parallel to lines of constant phase. For waves that are forced at the surface, we need to choose the sign in eq. (1.4) to ensure that the group velocity points upwards, or equivalently the phase propagation points downward.

Two examples: For large k and small m, the phase lines are almost vertical, and group velocity is also vertical. From the continuity equation, u/w=m/k, so the parcel movements are almost vertical, ω is close to N. For small k and large m, the phase lines are almost horizontal, so is the group velocity. Since the parcel movements are almost horizontal, the restoring forcing is weak and ω is close to zero.

The atmosphere is compressible. Inclusion of compressibility in our equations will give rise to sound waves, which will couple with the gravity waves. This leads to additional modes. However, if phase speeds of the gravity waves are low compared to the speed of sound and the vertical scale of the wave is smaller than the scale height, then the above expression can still be used.

Many processes can excite gravity waves: thunderstorms, winds blowing over the ocean. We will briefly look at topographic waves.

If we have uniform flow u over infinitesimal sinusoidal topography with wavenumber k, it is equivalent to forcing the atmosphere with a frequency of uk. Can be deduce what would happen?

If uk is less than N, then m^2 is positive and the wave can propagate vertically.
If uk is larger than N, then m^2 is negative, and the wave is evanescent in the vertical.

In reality, background horizontal flow and stratification vary with height and mountains are isolated instead of periodic. So a wide variety of responses can occur. If background horizontal flow and stratification vary slowly compared to the wavelengths of the waves, the problem is that of wave propagation in inhomogeneous medium and is analogous to that of geometric optics. For example, when waves propagate into a region of higher N, m becomes larger, i.e. the wave will be compressed in terms of its vertical scale. Variations in the mean winds also affect the propagation of the waves.

There are two central points that we want to get across. The first is that without rotation, gravity waves are very efficient in eliminating horizontal density gradient or temperature gradient in a stably stratified fluid. A simple example of this is that water surfaces (without winds) are flat, so that pressure is constant on a constant height surface.

We will now play a movie in which the atmosphere is initially warmer at the center. We see that gravity wave fronts spread out over a large distance, which is roughly the distance over which gravity waves are dissipated. Temperature anomalies that span the depth of the troposphere can travel at 50m/s and go around the globe in 10 days (4.e7m/50m/s). This is much more efficient than diffusion, even eddy diffusion. We will also see waves with greater vertical scales propagate faster, as can also be seen from the dispersion relation. This may be understood this way: the effective density difference over a greater vertical scale is greater so the reduced gravity is greater than that over a shallower vertical scale.

Now let's look at some observations. What is plotted is the geopotential height. Given the hydrostatic equation and use the ideal gas law, we have

$$d\Phi = gdz = -\frac{dp}{\rho} = -RTd\ln p$$

Integrate, we have

$$g_0(Z_2 - Z_1) = \Phi(z_2) - \Phi(z_1) = R\int_{p_2}^{p_1} Td\ln p$$

$$Z = \Phi(z)/g_0$$

This is called the hypsometric equation. For our purpose, we can neglect variations in g so that the geopotential height is the same as height z. Since in hydrostatic equilibrium, there is a one-to-one correspondence between p and Z, one may use either as the vertical coordinate. In meteorology, p is often used. Among the many reasons, pressure is easier to measure, and also has some advantage when used as a coordinate in the place of z. Pressure gradient force can be written in terms of geopotential height gradient:

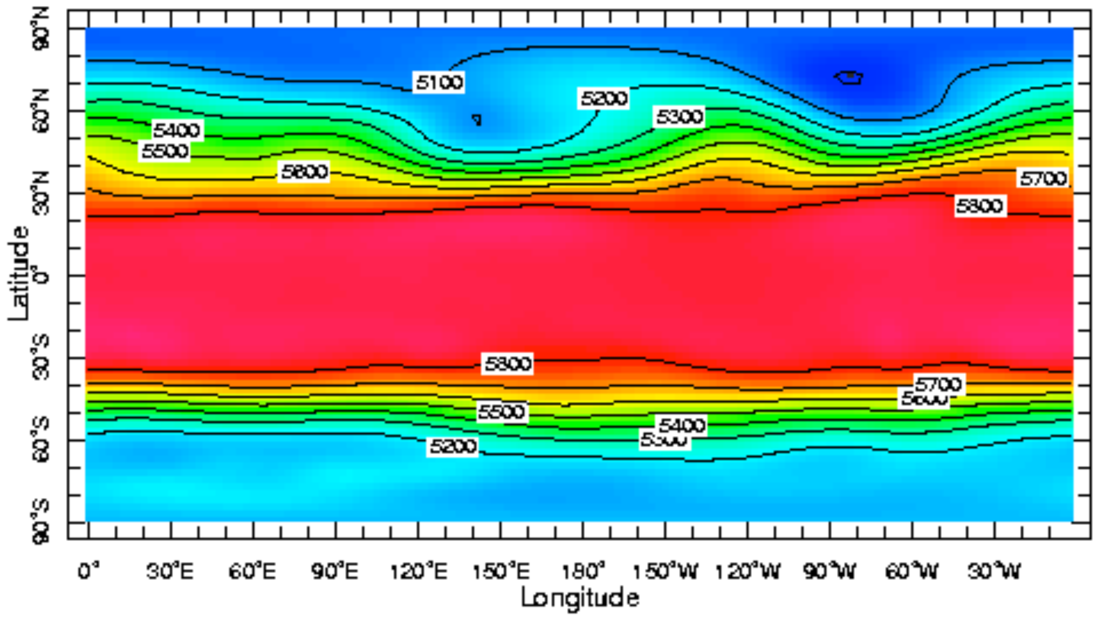
$$dp = \left(\frac{\partial p}{\partial x}\right)_z dx + \left(\frac{\partial p}{\partial z}\right)_x dz$$

$$= \left(\frac{\partial p}{\partial x}\right)_z dx + \left(\frac{\partial p}{\partial z}\right)_x \left(\left(\frac{\partial z}{\partial x}\right)_p dx + \left(\frac{\partial z}{\partial p}\right)_x dp\right)$$

$$0 = \left(\frac{\partial p}{\partial x}\right)_p = \left(\frac{\partial p}{\partial x}\right)_z + \left(\frac{\partial p}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_p$$

$$\left(\frac{\partial p}{\partial x}\right)_z = g\rho\left(\frac{\partial z}{\partial x}\right)_p$$

We have used the hydrostatic relation in the last step. So, a high Z on a constant pressure surface implies a high p on a constant Z surface.



Pressure 500 mb Time Jan

We see that the geopotential height as constant pressure is pretty much flat in the tropics where the effect of rotation is small.

Question: Given the 500mb Z and that the tropics is warmer, do we expect the Z difference between the pole and the equator to be greater or smaller near the surface?

The second main point is that momentum and energy fluxes are associated with the propagation of gravity waves (and waves in general). If they can propagate over a long distance before they are dissipated, they can transfer momentum and energy fluxes over a long distance. It is key to understanding mesosphere circulation. The momentum drag imparted on the flow when gravity waves break explains the peculiar fact that it is warmer in the winter hemisphere in the mesosphere. There are many other such examples.

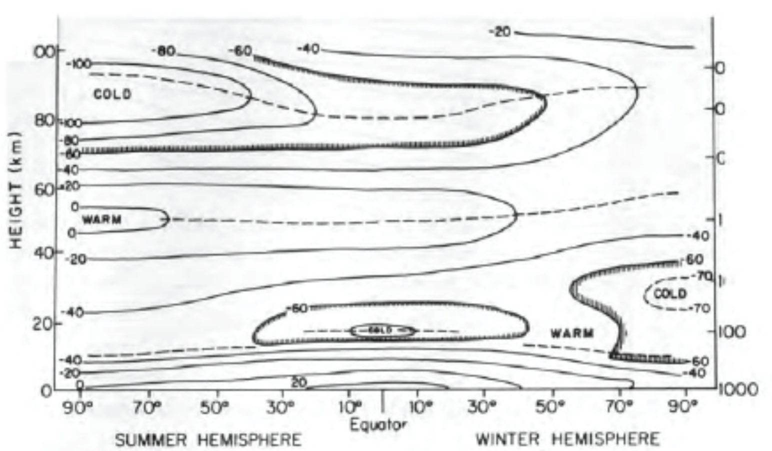


Fig. 1.3. Schematic latitude-height section of zonal mean temperatures (°C) for solstice conditions. Dashed lines indicate tropopause, stratopause, and mesopause levels. (Courtesy of R. J. Reed.)