The global energy balance

If we look at Earth from a distance as a planet orbiting a star, can we deduce its temperature? This is relevant to e.g. determining whether there is liquid water on the planet, or whether the planet is habitable. To do so, let us examine Earth's energy balance. Consider the first law of thermodynamics:

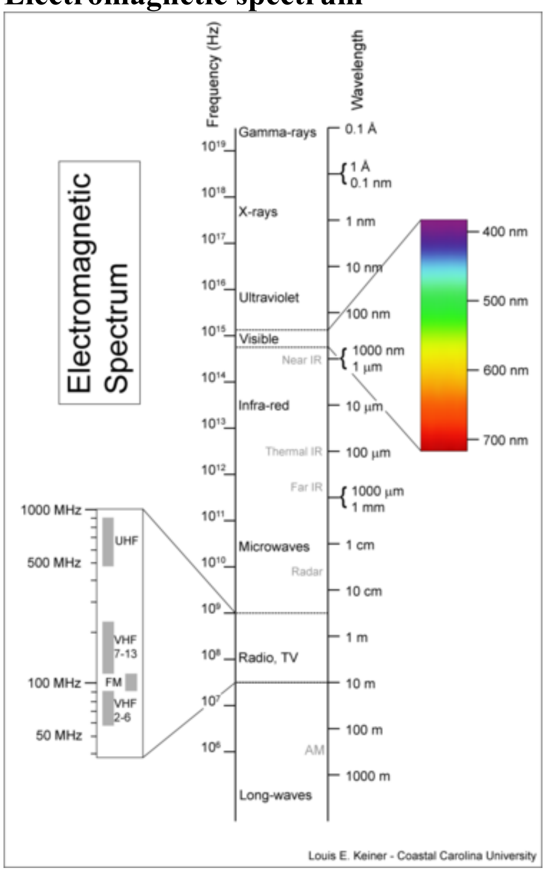
$$dU = \delta Q + \delta W \tag{1}$$

where dU is the change in the internal energy of the system, δQ is the amount of heat added, and δW is the work done to the system.

The work done to Earth by its environment (δW) is negligible, thus we need δQ=0 for Earth to be in energy balance (dU=0)[1].

Heat exchange can occur in the form of conduction, convection, and radiation. The heat exchange here is almost entirely in the form of radiation. To be in energy balance, energy received from the sun's radiation needs to be balanced by Earth's radiative emission.

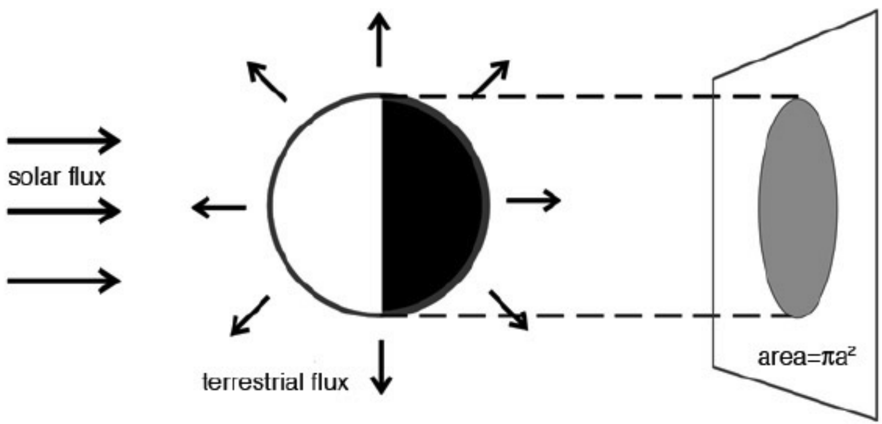
Electromagnetic spectrum



[1] This doesn't have to be true. Giant planets such as Jupiter and Saturn lose more heat that they absorb with ratios of 1.7 and 1.9 for Jupiter and Saturn, respectively. The difference is due to gradual loss of the accretion energy. The geothermal heat on Earth is ~0.1W/m^2, and may be neglected for the present purpose.

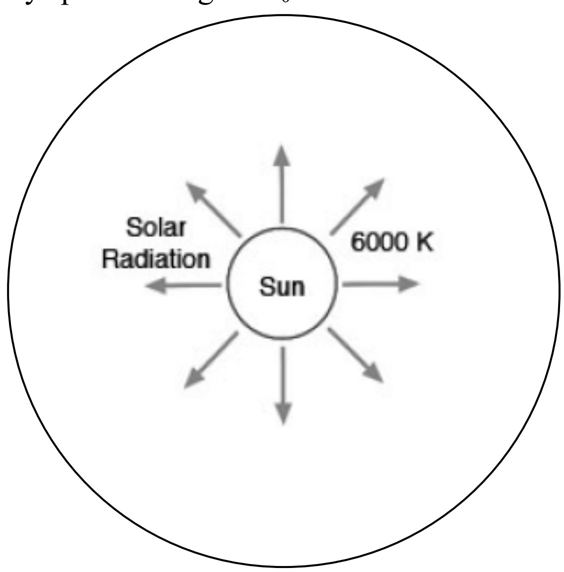
Define wavelength λ, frequency ν, and $\nu\lambda=c$, the speed of light. Often people also use wavenumber ($1/\lambda$) instead of frequency.

Energy received from the sun:



What is the energy flux density at the mean distance of Earth from the sun (1.5×10^{11}m)?

The solar *luminosity* (total energy flux from the sun) $L_0=3.9\times10^{26}$W. Very little of this energy is lost in space, which is effectively a vacuum. Thus integrating the energy flux density over any sphere will give L_0.



The flux density at Earth distance is therefore (assuming it is uniform over the sphere)

$$S_d = \frac{L_0}{4\pi d^2}$$

This is called the solar constant. Despite its name, it's clear it varies with the distance from the Sun. For Earth, it is S_0=1361W/m^2. Note that this number was revised down by ~5W/m^2 in 2011[2] (!). Even for Earth, solar "constant" varies by ~0.1% over a solar cycle (11 years). It also varies as the star evolves. Early in the lifetime of Earth, the solar luminosity was ~30% lower.

The energy flux intercepted by Earth is therefore $\pi R^2 S_0$, where R is the radius of Earth. Not all energy flux intercepted by Earth is absorbed (converted to energy of Earth); some are reflected. Define planetary albedo: α=energy reflected/energy intercepted, and we have:

$$\text{Absorbed solar radiation} = S_0(1-\alpha)\pi R^2 \tag{2}$$

Earth's albedo is ~30%. Note this albedo is averaged over the globe and over all wavelengths. An important contributor to this is cloud. We can view Earth's albedo with and without clouds at the following website:
http://iridl.ldeo.columbia.edu/SOURCES/.NASA/.ERBE/.Climatology

How much energy does Earth radiate away? It is a good approximation to assume that Earth radiates like a blackbody (an object that absorbs all radiation incident on it. BUT we just said Earth only absorbs 70% of solar radiation! We will come back to this in just a moment).

Basic radiometric quantities:

To describe a radiation field, we need to know the rate of energy flow at any given point, in any given direction, and at any given frequency. For that, we define:

Monochromatic intensity (or monochromatic radiance)
The amount of energy within a unit frequency (wavelength) interval that flows within a unit solid angle of a particular direction through a unit plane surface area perpendicular to this direction in a unit time interval

$$I_v = \frac{dE_v}{\cos\theta dA d\omega dv dt} \tag{3}$$

A solid angle ω is defined as the ratio of the area of a spherical surface to the square of the radius, and has the unit of steradian. A solid angle is an extension of angle to 3-dimensions. In polar coordinates, the differential solid angle is $d\omega=\sin\theta d\theta d\varphi$, where φ is the azimuthal angle and θ is the zenith angle.

The *monochromatic intensity* (or radiance) of a blackbody is given by the Planck's function:

[2] For more details, see: Kopp, G., and J. L. Lean (2011), A new, lower value of total solar irradiance: Evidence and climate significance, *Geophys. Res. Lett.*

$$B_\nu(T) = \frac{2\nu^2}{c^2}\frac{h\nu}{e^{h\nu/kT} - 1} \tag{4}$$

where h is the Planck's constant, k is the Boltzmann's constant, c is the speed of light, ν is the frequency of radiation, T is the temperature in Kelvins. A table of these constants and more are included in this note.

The Planck's function was first derived by empirically connecting two earlier formulas: Rayleigh-Jeans' distribution ($\lambda \to \infty$ or $\nu \to 0$) and Wien's distribution ($\lambda \to 0$ or $\nu \to \infty$). In order to provide theoretical justification for this formula, Planck hypothesized that the emitted energy is quantized, marking the beginning of quantum physics.

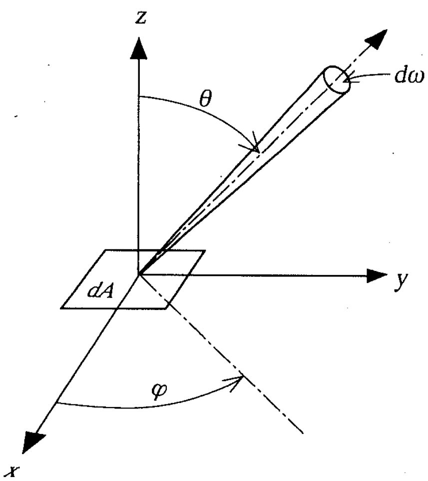


Fig. 3.1 Diagram showing the angles that define the radiance flowing through a unit area dA in the x–y plane, in the direction defined by the zenith angle θ, and the azimuth angle φ, and within the increment of solid angle $d\omega$.

Fig. 3.1 of Hartmann

Monochromatic flux density (or monochromatic irradiance)
The amount of energy within a unit frequency (wavelength) interval that flows through a unit plane surface area with a specified orientation in a unit time interval.

As energy can flow through a surface through different directions, the flux density is related to intensity by

$$F_\nu = \int_{hemisphere} I_\nu \cos\theta \, d\omega \tag{5}$$

In polar coordinates, we have

$$F_\nu = \int_0^{2\pi} d\varphi \int_0^{\pi/2} I_\nu \cos\theta \sin\theta \, d\theta \tag{6}$$

If radiation is isotropic (I_ν independent of angle), the integration over all angles in a hemisphere gives $F_\nu = \pi I_\nu$. This is a directional flux. For example, for a horizontal surface, if we integrate the upper hemisphere, we get the *upward flux* through this surface, and if we integrate the lower hemisphere, we get the *downward flux* through this surface. One could also integrate the whole sphere, and the result is the *net flux*. (If one integrates over the whole sphere without considering the zenith angle, one gets the actinic flux: $F_{actinic} = \int_0^{4\pi}\int_0^{\infty} I_\nu \, d\nu \, d\omega$. This is proportional to the number of photos passing a point and is important to photochemistry.)

When I_ν and F_ν are integrated over all frequencies (or over a finite frequency interval), they are called the intensity I and flux density F.

The above discussion can be equally made in terms of wavelength and because

$$\int_{\nu}^{\nu+d\nu} I_\nu \, d\nu = \int_{\lambda}^{\lambda+d\lambda} I_\lambda \, d\lambda$$

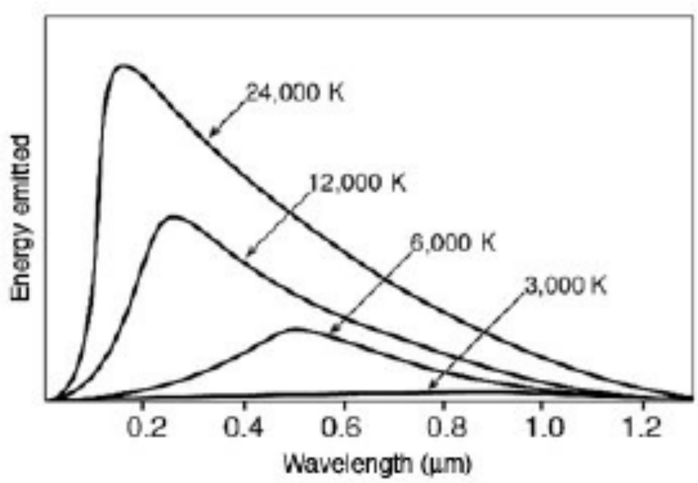
and

$$\nu = \frac{c}{\lambda}$$

$$d\nu = -\frac{c}{\lambda^2} d\lambda$$

We have $I_\lambda = \frac{c}{\lambda^2} I_\nu$ (the minus sign is canceled from reversing the direction of the integration).

Here is a plot of Planck's functions at a few different temperatures.



Constants and Conversions for Atmospheric Science

Universal constants

Universal gravitational constant	G	=	6.67×10^{-11} N m^2 kg^{-2}
Universal gas constant in SI units	R^*	=	8.3143 J K^{-1} mol^{-1}
Gas constant in chemical units	$(R_c)^*$	=	0.0821 L atm K^{-1} mol^{-1}
Speed of light	c	=	2.998×10^8m s^{-1}
Planck's constant	h	=	6.626×10^{-34} J s
Stefan-Boltzmann constant	σ	=	5.67×10^{-8} J s^{-1} m^{-2} K^{-4}
Constant in Wien's displacement law	$\lambda_{max}T$	=	2.897×10^{-3} m K
Boltzmann's constant	k	=	1.38×10^{-23} J K^{-1} molecule^{-1}
Avogadro's number	N_A	=	6.022×10^{23} molecules mol^{-1}
Loschmidt number	L	=	2.69×10^{25} molecules m^{-3}

Air

Typical density of air at sea level	ρ_0	=	1.25 kg m^{-3}
Gas constant for dry air	R_d	=	287 J K^{-1} kg^{-1}
Effective molecular mass for dry air	M_d	=	28.97 kg kmol^{-1}
Specific heat of dry air, constant pressure	c_p	=	1004 J K^{-1} kg^{-1}
Specific heat of dry air, constant volume	c_v	=	717 J K^{-1} kg^{-1}
Dry adiabatic lapse rate	g/c_p	=	9.8×10^{-3} K m^{-1}
Thermal conductivity at 0°C (independent of pressure)	K	=	2.40×10^{-2} J m^{-1} s^{-1} K^{-1}

Water substance

Density of liquid water at 0°C	ρ_{water}	=	10^3 kg m^{-3}
Density of ice at 0°C	ρ_{ice}	=	0.917×10^3 kg m^{-3}
Gas constant for water vapor	R_v	=	461 J K^{-1} kg^{-1}
Molecular mass for H_2O	M_w	=	18.016 kg kmol^{-1}
Molecular weight ratio of H_2O to dry air	ε	=	$M_w/M_d = 0.622$
Specific heat of water vapor at constant pressure	c_{pw}	=	1952 J deg^{-1} kg^{-1}
Specific heat of water vapor at constant volume	c_{vw}	=	1463 J deg^{-1} kg^{-1}
Specific heat of liquid water at 0°C	c_w	=	4218 J K^{-1} kg^{-1}
Specific heat of ice at 0°C	c_i	=	2106 J K^{-1} kg^{-1}
Latent heat of vaporization at 0°C	L_v		2.50×10^6 J kg^{-1}
Latent heat of vaporization at 100°C			2.25×10^6 J kg^{-1}
Latent heat of sublimation (H_2O)	L_s		2.85×10^6 J kg^{-1}
Latent heat of fusion (H_2O)	L_f		3.34×10^5 J kg^{-1}

Earth and Sun

Acceleration due to gravity at sea level	g_0	=	9.81 N kg^{-1}
Mass of the Earth	$m_\oplus$	=	5.97×10^{24} kg
Mass of the Earth's atmosphere	m_e	=	5.3×10^{18} kg
Radius of the Earth	R_E	=	6.37×10^6 m
Area of the surface of the Earth		=	5.10×10^14 m^2
Mass of an atmospheric column	m_a	=	1.017×10^4 kg m^{-2}
Atmosphere to Pascals	1 atm	=	1.01325×10^5 Pa
Rotation rate of Earth	Ω	=	7.292×10^{-5} s^{-1}
Mass of the sun	$m_\odot$	=	1.99×10^{30} kg
Radius of the sun	$r_\odot$	=	6.96×10^8 m
Mean earth-sun distance	d	=	1.50×10^{11} m = 1.00 AU
Solar flux	E_s	=	3.85×10^{26} W
Average intensity of solar radiation	I_s	=	2.00×10^7 W m^{-2} sr^{-1}

Units and Conversions

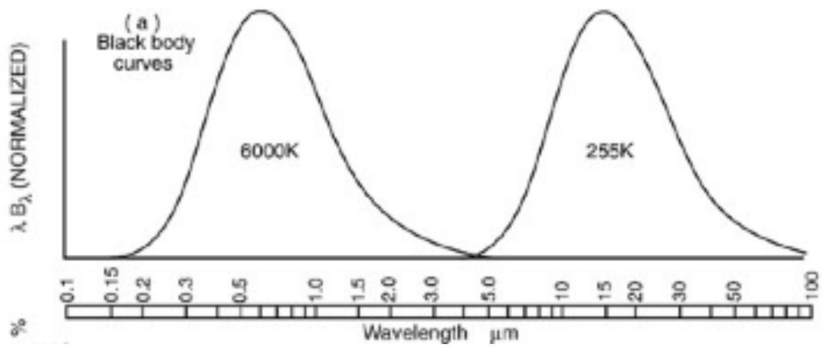
Fahrenheit-Celsius conversion	T_C	=	$\frac{5}{9}(T_F - 32)$
Kelvin-Celsius conversion	T_K	=	$T_C + 273.15$
Hectopascal conversions	1 hPa	=	1 mb = 10^3 dynes cm^{-2}
Cubic meters to liters	1 m^3	=	1000 L
Days to seconds	1 d	=	86,400 s
Calories to Joules	1 cal	=	4.1855 J
Latitude conversions	1° lat	=	60 nautical mi = 111 km = 69 statute mi
Longitude conversions	1° lon	=	111 km × cos(latitude)
Knots to miles per hour	1 knot	=	1 nautical mi/h = 1.15 statute mi/h
Meters per second to knots	1 m s^{-1}	=	1.9426 kt
Sverdrups to m^3 s^{-1}	1 Sv	=	10^6 m^3 s^{-1}
Dobson unit	1 DU	=	2.6×10^{16} molecules O_3 cm^{-2}

Note that the wavelengths with the peak emission intensity decrease with temperature. This can be quantified by requiring $\partial B/\partial \nu = 0$, which gives the Wien's displacement law (you are encouraged to try the derivation yourself).

$$\lambda_m T = const$$

$$T / \nu_m = const$$

Solar radiation peaks at ~0.6 microns, corresponding to a temperature of 6000K (This is one way to find out the temperature of a star). A planet like Earth is a lot colder, therefore emits at much longer wavelengths.



Indeed, with a reasonable temperature for Earth, there is little spectral overlap between its emission and that of the sun. So Earth can be bright (reflective) for the solar radiation but black at wavelengths where it emits most of its energy. Snow is one such example. It is very bright in the visible, but very dark (absorptive) at wavelengths of terrestrial radiation. So it is okay to assume Earth as a blackbody even though it reflects 30% of sunlight: reflection and absorption of a material depends on the wavelength under consideration.

In atmospheric science and climate research, the term *shortwave* refers to wavelengths less than 4 μm, which contain most of solar radiation, and the term *longwave* refers to wavelengths longer than 4 μm, which contain most of the terrestrial radiation.

Integration of the Planck's function over all frequencies and all angles gives the Stefan-Boltzmann law:

$$\begin{aligned} F &= \int_{hemisphere} \cos\theta d\omega \int_0^\infty B_\nu \, d\nu \\ &= \pi \int_0^\infty B_\nu \, d\nu = \sigma T^4 \end{aligned} \qquad (7)$$

where σ is the Stefan-Boltzmann constant and

$$\sigma = \frac{2\pi^5 k^4}{15 c^2 h^3}.$$

As blackbody radiation is isotropic (independent of angle), the integration over all angles in a hemisphere gives the factor π. The integration over frequency is not entirely trivial but can be done with some mathematical tricks. A derivation using contour integral can be found here: http://en.wikipedia.org/wiki/Stefan-Boltzmann_law#Appendix

Now, we have:

$$\text{Emitted radiation by Earth} = 4\pi R^2 \sigma T^4 \qquad (8)$$

Now we are ready to use energy balance to estimate Earth's temperature. Equating Eq. (2) and Eq. (8), we have:

$$T_e = \left[\frac{S_0 (1-\alpha)}{4\sigma} \right]^{1/4} \qquad (9)$$

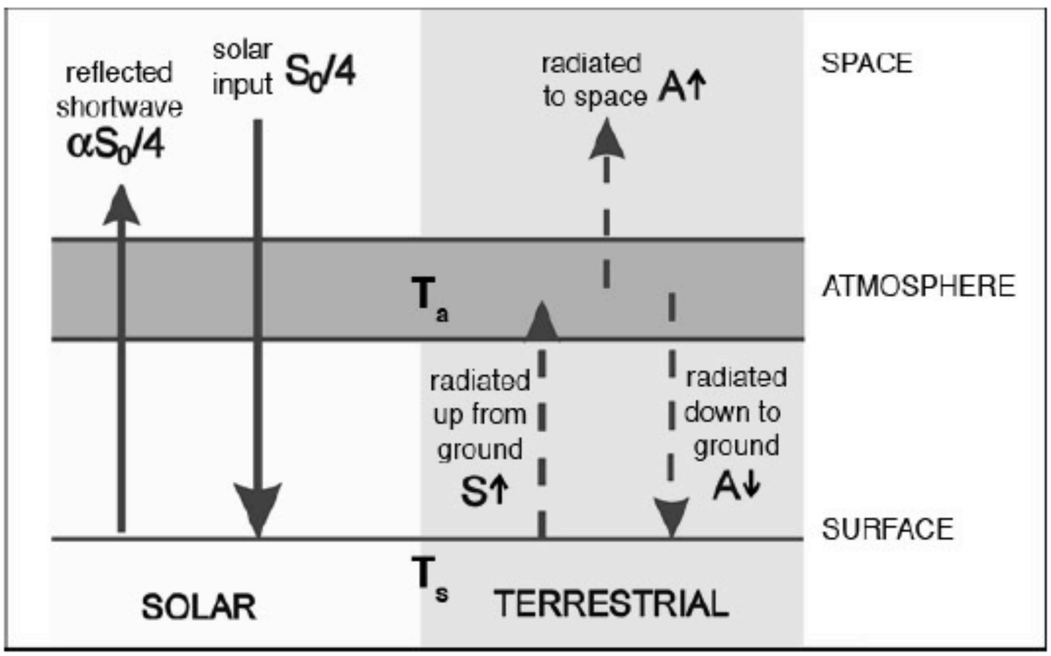
This is called the emission temperature. Note that the emission area is the surface of the sphere while the area that Earth intercepts sunlight is that of a disk. This gives the factor 4. There is no dependence on R. What we have just accomplished is quite significant: we estimated the temperature of a planet based on the luminosity of the star, planet-star distance, and the albedo of the planet.

Plug in the numbers, we have T=255K or -18C.

The observed global mean surface temperature is ~288K or 15C (You can read more about how this is observed at http://www.cgd.ucar.edu/cas/tn404/text/tn404_1.html). What went wrong?

The Greenhouse effect:

We neglected that Earth has an atmosphere, which interferes with the radiation so we need to modify our calculation. We will have a more formal discussion of radiative transfer later. For the moment, let us consider the following simple case, where we assume the atmosphere is a homogeneous layer transparent to solar radiation but opaque to terrestrial radiation. This is possible, again because the two occupy very different wavelengths.



Consider the energy balance at the top of the atmosphere, we get the same result as Eq. (9): $T_a = T_e = \left[\frac{S_0(1-\alpha)}{4\sigma}\right]^{1/4}$

Now consider the energy balance at the surface. The surface now receives both the solar radiation and the radiation from the atmosphere: $A = \sigma T_a^4 = \frac{S_0(1-\alpha)}{4}$. The latter is absorbed by the surface, as it is approximately a blackbody for terrestrial radiation. Now we have

Absorbed solar radiation + radiation from the atmosphere=radiation from the surface

i.e. $$S_0(1-\alpha)/4 + A = \sigma T_s^4$$

so:

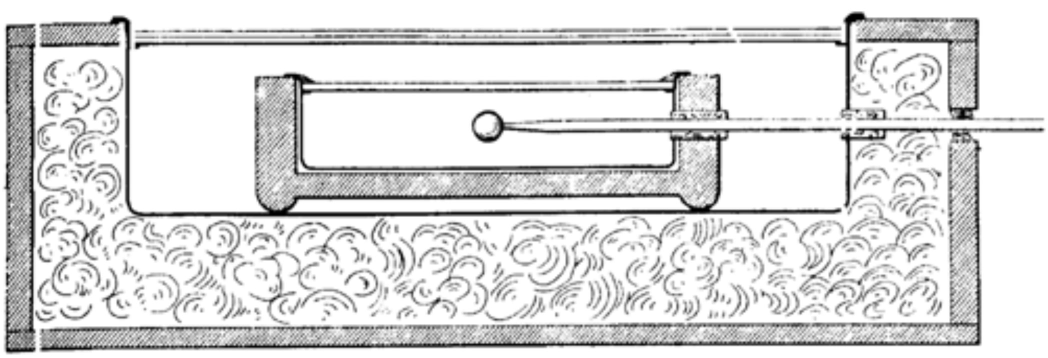
$$T_s = 2^{1/4} T_a = 303K$$

This is the greenhouse effect, which in this case warms the surface by 48K!

If we have N opaque atmosphere layers, we have $T_s = (1+N)^{1/4} T_e$

A solar cooker effect?

Glass made of silicate absorbs infrared radiation. Horace de Saussure made an apparatus that makes use of the effect that we just talked about in 1767 (To read more, see this link http://solarcooking.org/saussure.htm). Today's solar cookers are based on this idea.



Cross-section of Langley's hot box, which was similar to de Saussure's later models. A thermomether penetrating the walls at right was used to measure the air temperature inside the inner box.

The controversy around the name "greenhouse effect": in many contemporary greenhouses, the cover is made of plastic, which is transparent to infrared radiation, and it works by insulating the greenhouse from wind, so as to reduce the lost of heat. So some consider "greenhouse effect" a misnomer, but it really depends on whether the greenhouse is built with glass or plastic.

A leaky greenhouse

The atmosphere is not a homogeneous layer, is not opaque to infrared radiation and is not a blackbody. Define monochromatic emissivity ε_v as the ratio of the monochromatic intensity of the radiation emitted by the body to the corresponding blackbody radiation

$$\varepsilon_v = \frac{I_v(emitted)}{B_v(T)}$$

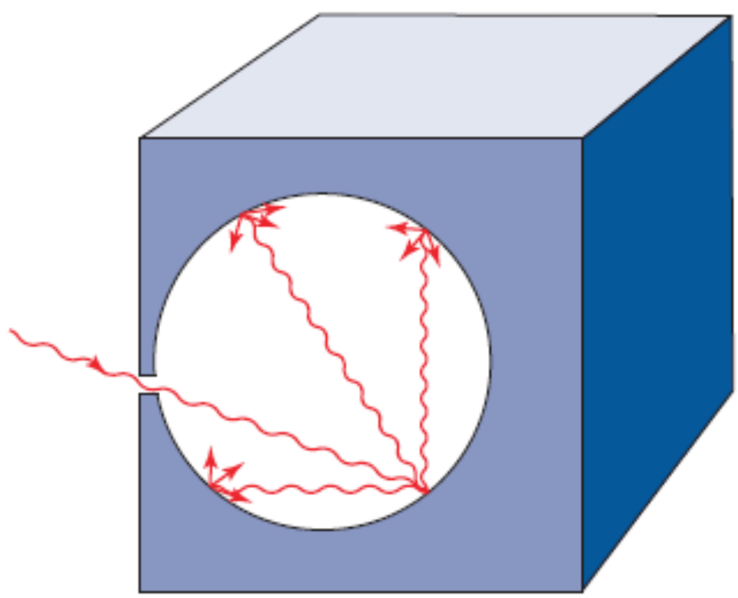
and the monochromatic absorptivity a_v

$$a_v = \frac{I_v(absorbed)}{I_v(incident)}$$

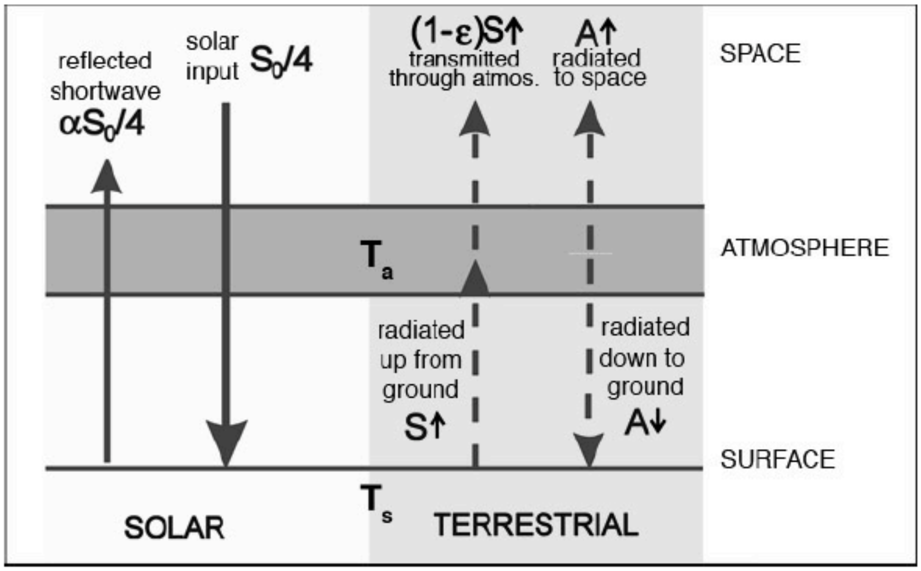
Kirchhoff's law:

$$\varepsilon_v = a_v \tag{10}$$

To understand the basis for Kirchhoff's law, let us consider a cavity with a very small aperture. While there is no blackbody in nature, radiation field in such a cavity approaches that of a blackbody. Place an object made of any material in the cavity. In equilibrium, the temperature of this object is the same as that of the wall. Otherwise, we would have built a perpetual machine and violated the second law of thermodynamics. The amount of radiation it absorbs should equal to the amount that it emits to maintain equilibrium of the radiation field so that absorptivity is equal to emissivity. For the special case of a blackbody that absorbs all radiation, it must emit the same radiation as that is in the cavity. This is why blackbody radiation is also called cavity radiation. Now, if we consider absorptivity and emissivity as intrinsic properties of matter, then the equality should hold even when the object is removed from the cavity. For gases, the last condition is satisfied when the frequency of molecular collisions (which maintains the Boltzmann distribution) is much larger than the frequency with which molecules absorb and emit radiation at the relevant wavelength. This condition is called local thermodynamic equilibrium (LTE). In Earth's atmosphere, LTE is satisfied below ~60km.



Now let us consider the energy balance with a leaky greenhouse, illustrated in the figure below.



Balance at the top of the atmosphere:

$$\frac{1}{4}(1-\alpha)S_0 = A\uparrow + (1-\varepsilon)S\uparrow$$

Balance at the surface:

$$\frac{1}{4}(1-\alpha)S_0 + A\downarrow = S\uparrow$$

Combining the two, one can also get the balance for the atmosphere. For a homogeneous layer in LTE, there is no difference between up and down, so $A\uparrow = A\downarrow$, and we have

$$S\uparrow = \sigma T_s^4 = \frac{1-\alpha}{2(2-\varepsilon)}S_0 = \frac{2}{2-\varepsilon}\sigma T_e^4$$

Balance for the atmosphere (and make use of Kirchhoff's law):

$$A\uparrow + A\downarrow = 2\varepsilon\sigma T_a^4 = \varepsilon T_s^4$$

$$T_a = \left(\frac{1}{2}\right)^{1/4} T_s$$

These toy models illustrate the basic concept of the greenhouse effect, but there are a number of issues: the atmosphere is of course not a homogeneous layer; we have all heard about global warming by CO_2, but its concentration is only a few hundred parts per million, why do we care about gases of such a small (and even smaller) concentrations? To address how radiation interacts with the atmosphere more generally and more rigorously, we need to know more about radiative transfer.