Math 168 Weekly assignment #1 Due Friday, Sep 14

About You

For this section only, if you prefer your responses to come just to the instructor, you may email your answers to whitney@math.harvard.edu any time prior to class on the 14th.

1. Let me know your name. (Well, all assignments should have your name on them. But I figured I would say that just this one time.)
2. What's your academic year?
3. What sparked your interest in *Making Math Material*? What do you hope to get out of the course?
4. Write three-ish paragraphs about your relationship with mathematics. How was that relationship formed? What's positive or negative about it? Be sure to mention your most and least favorite math classes in your life and something about why.
5. What related courses did you take last year or are you taking now?
6. Scheduling. Please let me know the times each week that you *cannot* make it to events related to this course, such as office hours or review sessions, and the times that are *difficult* to make but which you could do if you felt you had to.
7. What else should I know about you?

About polyhedra

This is an example of a "theoretical" assignment in this class. Here are the ground rules for all such assignments: You should write clear, convincing explanations and arguments for the points you make. Your argument should be understandable and convincing to a friend with a similar math background to you. There is not a need to use formal mathematical notation or language, but if you are most comfortable expressing yourself that way and you can do so without sacrificing clarity of exposition, it is OK to do so. Finally, all of your responses must be accompanied with an explanation or justification. In other words, for question 3, "Yes." or "No." is not an acceptable answer; you need to argue why your answer is correct.

1. Based on the discussion we had in class, write a definition of what a polyhedron is as clearly, precisely, and comprehensively as you can. Then explain why it is (or isn't!) a good definition – some aspects of "good" are "does it capture the notion we are trying to understand?" and "does it clarify the details of this notion?" and "will it be easy/useful to work with in figuring out things about this notion?"
2. Now define the *faces*, *edges*, and *vertices* of a polyhedron.
3. Must any polyhedron have all of these elements? If so, explain why. Does it follow from your definition?
4. What is the *minimum* number of each of these elements a polyhedron can have? (For this one problem *only*, no explanation is required.)
5. What is the minimum number of faces that can meet at a given vertex?
6. Call a vertex of a polyhedron *convex* if there is some plane through that vertex such that the entire polyhedron lies on one side of that plane. Argue that there must be at least one convex vertex in any polyhedron.

7. Consider all of the faces that meet at a convex vertex. Each face has some *vertex angle* there (just the ordinary interior angle of the face at that vertex). What can you say about the sum for all of these faces of their vertex angles at that vertex?
8. We are interested in regular polyhedra. The first property that makes a polyhedron regular is that all of its vertices are congruent, which means that if you intersect the polyhedron with spheres of the same radius at any two vertices, where that radius is small enough that the spheres do not intersect any other vertices or edges or faces of the polyhedron (other than the ones meeting at that vertex), then the two pieces sliced off by these intersections are identical. Can a regular polyhedron have any non-convex vertices?
9. The second property that makes a polyhedron regular is that all of its faces are the same regular polygon (an equilateral triangle, square, or so on..). In light of your answers to questions 5, 7, and 8, which regular polygons can occur as the faces of a regular polyhedron?
10. For each of the polygons in question 9, what are the possible numbers of that kind of polygon that might meet at a vertex of the polyhedron?
11. We will take it as granted that two regular polyhedra with the same type of face and the same number of that type of face meeting at each vertex are similar (we defined *similar* in class). So what is the maximum possible number of different regular polyhedra? (Note: it would remain to argue that each of the logical possibilities enumerated so far actually exist – that there really is a way of piecing together a polyhedron of that type. Hopefully the dice we used in class are convincing of that! So you don't need to address this issue in your answer.)

Challenge problems

For theoretical challenge problems, you should use a greater level of mathematical formalism and provide reasonably formal proofs where appropriate. You can do any, all, or none of the challenge problems; each successful effort on a problem will reduce the weight of all of the other elements of the course.

1. Prove rigorously the first statement in question 11 above.
2. Define the *loop number* of a convex polyhedron P as the minimum number of distinct copies of $P, P_1 \ldots P_n$, necessary to produce a configuration such that (i) each P_i is connected face-to-face to P_{i-1} and P_{i+1}, where P_0 is understood as P_n and P_{n+1} is understood as P_1, and (ii) each P_i is disjoint from all other P_j except for these two adjacent ones. The loop number is considered to be infinity if it not possible to make such a loop regardless of the number of P_i used. Determine the collection of possible loop numbers of convex polyhedra.
3. The remaining problems outline a proof that the loop number of the regular tetrahedron T is infinity, as claimed in class. First show that if $V_1 \ldots V_4$ are the four vertices of T, then every point in $\mathbf{R}^3$ can be uniquely expressed in the form $x_1V_1 + x_2V_2 + x_3V_3 + x_4V_4$ where the x_i sum to 1. Hence we may identify $\mathbf{R}^3$ with the space of 4-dimensional vectors $\mathbf{x}$ whose coordinates sum to 1; all vectors referred to below are of this form.
4. Show that for each reflection r_i in the face of T opposite V_i there is a matrix $\mathbf{M}_i$ such that r_i takes each $\mathbf{x}$ to $\mathbf{xM}i$ (ordinary vector-matrix multiplication) and determine each of the $\mathbf{M}_i$ exactly.
5. Prove that if the loop number of the tetrahedron is finite, then there is a sequence of matrices $\mathbf{N}_1..\mathbf{N}_k$ such that each $\mathbf{N}_j$ is one of the $\mathbf{M}_i$, $\mathbf{N}_j$ and $\mathbf{N}_{j+1}$ are distinct, and the product $\mathbf{N}_1\mathbf{N}_2 \ldots \mathbf{N}_{k-1}\mathbf{N}_k$ is a permutation matrix (every entry is 0 or 1 with a single 1 in each row and column).
6. Prove that if you replace each non-integer entry of each of the $\mathbf{N}_j$ with the same variable z, then there is a row in the result of the above product consisting of polynomials in z all of which have leading coefficient 1 and with one unique polynomial in that row of maximal degree.
7. Put the elements in problems 3 through 6 together to prove that the loop number of T is infinity.