Math 168 Weekly assignment #5

About vertex-to-vertex and edge-to-edge arrangements of cubes

In this assignment, you will pick **one** of several building options designated in **bold** below. For any build plan, your budget is 250 boxes, chosen from among the eight colors we saw in class: white, hot pink, red, orange, yellow-green, teal, purple, and gold. In addition to any specific written questions about the option you choose, you will make a software model or models of the structure you are going to build (and submit the save files) and you will prepare detailed build plans for your model for a team of two people, listing the inventory you need, and the steps to produce the final structure. As before, keep in mind the need to utilize both team members and to complete the build in 75 minutes. Also include a description of what mathematical aspect of the structure your use of color highlights or corresponds to. When you come to class on Friday, Oct 12, the materials will already be laid out. As soon as you have chosen a partner to work with and decided which of your two plans you mutually want to build, you should start immediately; no need to wait for further instruction or the "official" start of class. You are encouraged to use the already-constructed units from last time, detaching them as necessary, to save time where possible.

As we discussed in class on Oct 5, in seeking a three-dimensional arrangement of cubes that extends through a significant region of space, it's helpful to focus on the innermost "pocket" among the cubes. This has to be a polyhedron with the property that at least two squares meet at every vertex. The fact that the exterior cubes cannot impinge on each other at least intuitively enforces the condition that this pocket be convex (no indentations, or more formally, the entire line segment between any two points in the pocket is also contained in the pocket). However, I do not have a proof of this condition, which leads to

Option I: Construct a configuration containing a non-convex pocket.

The polyhedron comprising the pocket should be clearly defined by the square faces of the cubes adjacent to the pocket and the edges of those cubes. That is, every vertex of the pocket must be a vertex of some cube, and every edge of the pocket must be an edge of some cube. The configuration should be rigid, or it should be the case that the pocket is non-convex in all states the configuration can physically reach. Warning: I am not sure this option is possible; but if you do find such a configuration it definitely qualifies as mathematically interesting and hence a worthy subject of a build in its own right.

In any case, if we do assume convexity, then there are two main possibilities: either we opt for maximal symmetry and insist that all vertices of the basic pocket are identical, or we say that any convex polyhedron composed of equilateral triangles, squares, and regular pentagons is OK.

In the former case, in class we enumerated the possibilities for the configuration at a vertex of the pocket, and what polyhedron they corresponded to, as follows.

2 square, 1 triangle: triangular prism (not as full use of three-dimensionality, since it is just a "thickened" version of a two-dimensional triangle); 2 square, 2 triangle: cuboctahedron; 3 square: cube; 3 square, 1 triangle: rhombicuboctahedron; 2 square, 1 pentagon: pentagonal prism (not full use of three-dimensionality); 2 square, 1 pentagon, 1 triangle: rhombicosidodecahedron.

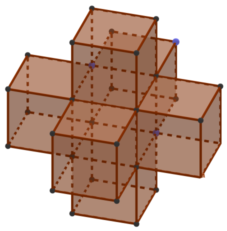
What we merely alluded to in class, but did not have time to explore, is how the configuration of cubes surrounding the initial pocket can be extended through a larger region of space. There are two basic possibilities I have identified, leading to

Option II: Construct a third mode of extending the initial configuration.

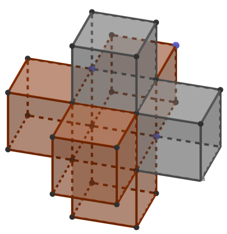
That is to say, if you come up with a systematic way of taking the basic configuration of cubes around any of the polyhedra above and extending it to cover a significant region of space, that does not fall into either of the categories (A) or (B) below, then building a model of the resulting extended configuration is inherently mathematically interesting. Warning: as with option I, it is not at all certain this is possible.

In any case, the two modes of extension I have identified are (A) using some of the cubes surrounding the initial pocket as part of the configuration surrounding another pocket, or (B) considering the outer "skin" of the configuration of cubes surrounding the inner pocket as a polyhedron itself and connecting those polyhedra face-to-face (or edge-to-edge). You can explore one of these possibilities in greater depth in one of the following options.

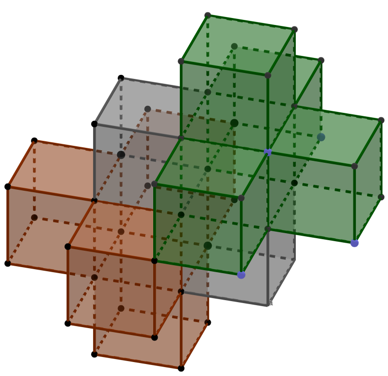
But first, here is an example of possibility (A) for cubical pockets. A cubical pocket looks like this:



(the space between all six cubes is hollow, which you can't quite see for sure in this rendering). But now, if you notice the two grey cubes at the upper right:

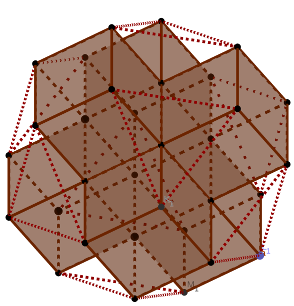


You can see that they could act as two of the cubes surrounding a second cubical pocket, so that we can add four more green cubes like so:

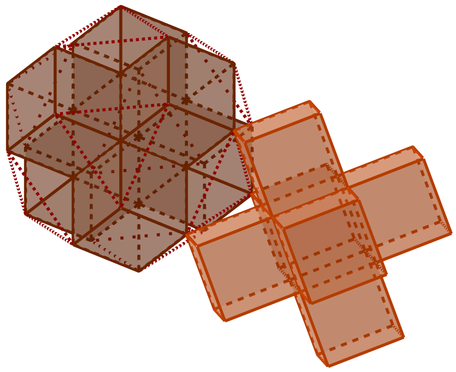


And of course now you can continue surrounding more cubical pockets in any direction you like.

For an example of possibility (B) using the same basic pocket, imagine adding additional edges around the outside of the basic unit, like so:



The outer polyhedron is analogous to a rhombicuboctahedron, but it is not regular: there are square faces with edge length 1, rectangles with edge length $\sqrt{2}$ and 1, and equilateral triangles of edge length $\sqrt{2}$, but otherwise the faces are arranged the same way as in a rhombicuboctahedron. Now you can take two such units and attach them so that two equilateral triangle faces coincide, like so:



and again, you can continue adding more units in this same fashion in any direction you like.

Option III: the cube-pocket shared lattice

In this option, you will explore the lattice created in the first example above.

1. Using a 3D plotting program or dynamic geometry software, such as Geogebra (under the top right menu you can select Views and then 3D Graphics), plot the centers of all of the boxes in the final diagram above, as well as all boxes that would share an edge with one of these if the pattern were extended. Connect two of the centers if the cubes centered at those points share an edge. Submit your Geogebra save file or a screen shot if you use some other program.

2. Identify the polyhedra formed by the edges you created in (1). There should be two types, and together they fill space with no gaps or overlaps, creating what's called a *honeycomb*, the three-dimensional analogue of a tessellation in two dimensions. How do you know you've correctly identified the polyhedra? You should be able to look up the mathematical properties of this honeycomb; submit a web link to a internet article about the honeycomb.

3. For your build, select any mathematically interesting subset of the honeycomb and model it using the edge-to-edge cubes from this cubical lattice. Note that the centers of the cubes correspond to the vertices of the honeycomb, and the edges of the honeycomb correspond to imaginary line segments joining the centers of adjacent cubes through the midpoint of the edge they share. Use similar criteria as for the face-to-face cube assignment for "mathematically interesting". Make sure to include enough of the honeycomb so that the resulting edge-to-edge cube structure seems like it will be reasonably rigid, given that each edge-to-edge joint can act as a hinge.

Option IV: the cuboctahedron-rhombicuboctahedron shared honeycomb

1. Create a model showing that if you share the cubes of a cuboctahedral pocket (or a rhombicuboctahedral pocket), the adjacent pocket naturally takes on the structure of a rhombicuboctahedral one (alternately, a cuboctahedral one). Submit your model.

2. If you extend this pattern, the cubes of the pattern together with the cuboctahedral and rhombicuboctahedral cavities form a honeycomb called the *cantellated cubic honeycomb*. You can look this up on line. In the overall honeycomb, what are the ratios of the numbers of cubes, cuboctahedra, and rhombicuboctahedra to each other?

3. Another useful characteristic of a cantellated cubic honeycomb is that if you take only the cubical cells and connect them vertex-to-vertex as in the honeycomb, the resulting structure is rigid (as long as it contains an entire cells of the other polyhedral types). Choose a subset of this honeycomb that you feel results in an attractive structure and model it and develop build instructions for it.

Option V: Rhombicosidodecahedron pocket face-to-face structures

Unfortunately, there is no uniform honeycomb which contains cubical and rhombicosidodecahedral cells. However, we can make a large variety of structures by connecting the outer polyhedra of such pockets formed by cubes, as in method (B) above. Keeping in mind that your cube budget allows a maximum of eight rhombicosidodecahedral pockets (each one takes 30 cubes), design and model an attractive structure composed of at least three such units. Create build instructions for your model.

Option VI: Cubic pocket face-to-face structures

1. Note that if you connect the rectangular faces of the outer polyhedron from the example for method (B) above, you will just be constructing a subset of the lattice from option III. Therefore, the only way to obtain a new structure with this type of connection is to connect equilateral triangular faces. Plot the **centers** of several (circa 20) of these units connected in a clump, and connect the centers that correspond to adjacent units with line segments. Submit your plot.

2. Do the vertices and edges you generated in (1) form a honeycomb? If so, what polyhedra are its cells? If not, what honeycomb are they a subset of?

3. Choose a mathematically interesting subset of the honeycomb in (2) and model it with triangle-to-triangle cubic pocket units. Submit the model and build instructions. Keep in mind your cube budget allows for a maximum of 40 such units.

Option VII: (Rhombi)cuboctahedral pocket face-to-face structures

Rhombicuboctahedral pocket units connected via their hexagonal faces produce the structure from option IV. However, any other means of connecting either cuboctahedral pocket units or rhombicuboctahedral pocket units would produce a new structure. For this option, choose one pocket type and one way of connecting them, and follow the outline of option VI for that combination.

Option VII: Lower-symmetry pockets

In addition to the cube, cuboctahedron, rhombicuboctahedron, and rhombicosidodecahedron, the following Johnson solids satisfy our criterion of having at least two squares at every vertex: J26, J27, J28, J29, J30, J31, J35, J36, J37, J38, and J39. (You can look up what these look like on line.)

1. Choose one of these Johnson solids. Create a model of cubes connected so that they form a pocket of precisely the shape of your Johnson solid. Submit that model.

2. Extend your model to explore what happens when you try to extend the configuration of cubes from (1) to extend through a region of space *entirely surrounding* that initial configuration, using either method (A) or (B) from above. Does there seem to be a pattern which will extend indefinitely through space? Why or why not? If so, what shapes are the other pockets in the resulting pattern, besides the intiial one you chose? Submit the extended model.

3. Develop build instructions for your model and submit them.

Challenge problem

Prove that there is no non-convex polyhedral pocket formed by vertex-to-vertex cubes, in the manner described in option I above. Warning: either this problem or option I is impossible, since they are direct logical negations of each other.