Math 168 Weekly assignment #4

About the Integer Lattice

1. Here is another way of analyzing the integer lattice in the plane: we can consider the plane to be the complex plane, with the horizontal axis real and the vertical axis imaginary. Then the lattice points are what are called the *Gaussian integers*: complex numbers of the form $a+bi$ for a, b integers. Notice that if you add, subtract, or multiply Gaussian integers (using the defining rule $i^2=-1$), the result is a Gaussian integer (no response needed on that point). The *conjugate* of a complex number $z=a+bi$ is $\overline{z}=a-bi$. The *norm* $N(z)$ of a complex number z is $z\overline{z}$, i.e., z times its conjugate. Show that (a) the norm of a complex number is just the square of its distance from the origin in the complex plane, (b) that the norm of a Gaussian integer is a positive (ordinary) integer, and (c) for any Gaussian integers v, w, we have $N(vw)=N(v)N(w)$.

2. Just as in the ordinary integers, a Gaussian integer may be prime, but we have to be careful with the definition, since any Gaussian integer $v=i(i^3v)$, and both i and i^3v are Gaussian integers. The troublemakers are the *units* in the Gaussian integers, namely Gaussian integers u such that for some other Gaussian integer v, $uv=1$. Then we say that a Gaussian integer $t=vw$ where v and w are both non-unit Gaussian integers is *composite*, and a *Gaussian prime* is a Gaussian integer that is not composite. (a) Determine all units of the Gaussian integers. (b) Show that if the norm of a Gaussian integer is prime (as an ordinary integer), then that Gaussian integer is a Gaussian prime.

3. Every ordinary integer n can be considered a Gaussian integer $n+0i$ as well. Is 2 considered as a Gaussian integer a Gaussian prime? What about 3?

4. It is a fact (which we will just accept without proof for our current purposes) that the Gaussian integers have a unique factorization property similar to that of the ordinary integers. Namely, any Gaussian integer t can be written as a product of Gaussian primes $t=q_1q_2\ldots q_n$, and any other expression of t as a product of Gaussian primes must have the same length, and can be re-ordered as $t=r_1r_2\ldots r_n$ such that each $r_i=u_iq_i$ where u_i is a unit. So, suppose p is prime as an ordinary integer but not prime as a Gaussian integer. How many prime factors does it have? What can you say about the relationship of those factors to each other? Can you say anything about the remainder when p is divided by 4?

5. Show that all Gaussian primes can be divided into two categories: Type I: ordinary integers which are prime in the Gaussian integers, and Type II: complex Gaussian integers with prime norms.

6. In light of the role of units in the Gaussian unique factorization property, say that Gaussian integers q, r are *associates* if there is a unit u such that $q=ur$. Show that among the Type II Gaussian primes, there is one value for the norm (what value?) so that any two primes of that norm are associates, and that for every other value of the norm there are exactly two Gaussian primes which are not associates of each other. What is the relationship between those two primes in the latter case?

7. Prove that if $w=a+bi$ is a Gaussian integer such that w^n is an ordinary integer for some integer n, then one of the following four possibilities must happen: $a=0, b=0, a=b$, or $a=-b$. (Hint: consider the prime factorization of w and split it up into three groups, consisting of the primes of Type I, and the two subtypes of primes of Type II described in (6). In the case of the pairs of non-associate primes that have the same norm, can you have just one of the pair in the prime factorization of w?)

About lattice polygons

1. For a complex number z, define $\theta(z)$ to be the angle (measured counterclockwise) from the positive horizontal axis in the complex plane to the ray connecting the origin to z. Show that $\theta(z_1z_2)=\theta(z_1)+\theta(z_2)$.

2. (Remember, we write τ for the angle measure of a full circle, $\tau=2\pi$.) Show that if the tangent of angle τ/n is rational, where n is a natural number, then n is either 1, 2, or 8. (Hint: you will need to find some way to use the previous two problems.)

3. In class we showed that there is no lattice equilateral triangle. Of course there is a lattice square, e.g., the unit square. Now determine all n for which there is a lattice regular n-gon (remember the method we used to prove there is no lattice equilateral triangle).

About lattice polyhedra

Since in this building cycle and the next we are interested in the geometry of cubes in three-dimensional space, it is natural to try to generalize the interesting facts we found out about the integer grid in two dimensions to three.

1. Make a table of lattice polyhedra, showing for each polyhedron its vertices (given by their integer coordinates), its volume, the number of lattice points in the interior of the polyhedron, and the number of lattice points on the surface of the polyhedron (including the edges and vertices). You should include at least five non-congruent examples, including at least three (not necessarily regular) tetrahedra, two different ones of which should have no interior lattice points and no surface lattice points other than the vertices. Your list overall should include polyhedra with at least two different numbers of faces.
2. Based on the data in (1), can you make a conjecture about the possible values for volumes of lattice polyhedra?
3. Prove your conjecture from (2). (If you are having trouble completing your proof, maybe you missed a possible type of volume. Go back to your list in 1 and add a tetrahedron with a base in the xy-plane and with the area of the base as small as possible, and with the fourth vertex having z-coordinate equal to 1, and try 2 and 3 again.)
4. In the section meeting, you learned about Pick's formula: the area of a lattice polygon in the plane is $I + B/2-1$, where I is the number of interior lattice points and B is the number of lattice points on the boundary (including the vertices). Looking at the data accumulated in (1), could there be a formula of the same type in three dimensions?

More about lattice polygons

1. Make a table of lattice polygons illustrating Pick's formula, showing the vertices, area, number of interior lattice points, number of boundary lattice points (including the vertices), and also the total number $\overset{1}{T}$ of lattice points (which is just the sum of the previous two columns), and the total number $\overset{2}{T}$ of lattice points (interior plus boundary) of the polygon produced by doubling every coordinate of every vertex of the given polygon. As in three dimensions, make sure your list contains at least five noncongruent polygons, at least three of which are triangles, and with at least two different numbers of sides overall.
2. Find an alternate version of Pick's formula giving the area of a polygon as a (degree-one polynomial) in terms of just $\overset{1}{T}$ and $\overset{2}{T}$. Make sure your formula works for all of your examples in the previous problem, but you are not required to provide a general proof of your formula.
3. What common fairly simple algebraic expression does the numerator of your formula in (2) remind you of, if you squint and imagine that the numbers above the 'T's were exponents (which they're not)?

More about lattice polyhedra

1. Intuitively, part of the issue in (4) in the first lattice polyhedra section above is that volume should be determined by three quantities, not just two. Now that we have the formula from (2) in the previous section, there is an obvious third quantity to add to the mix: $\overset{3}{T}$, the total number of lattice points (including on the surface and inside) of the polyhedron formed by tripling the coordinates of every vertex. Add columns for $\overset{1}{T}$, $\overset{2}{T}$, and $\overset{3}{T}$ to your table of polyhedra from (1) in the first polyhedra section.

2. Find a formula for the volume of a lattice polyhedron (as a degree-one polynomial) in terms of just the three quantities from the preceding problem. (Hint: problem (3) from the previous section.) Make sure your proposed formula works for all of the examples in your table, but no proof is necessary for this problem.

The moral of the story of this assignment: it matters what form you state a theorem in when you try to generalize it! That's one of the reasons there's value in trying to look at already known facts from different directions, something that building physical models can help with.

Challenge problems

1. Prove the formula in (2) in the final section above.
2. Determine all of the regular lattice polyhedra.
3. Find a formula for the number of lattice points on the surface of a lattice polyhedron in terms of $\overset{1}{T}$, $\overset{2}{T}$, and $\overset{3}{T}$.
4. If including $\overset{3}{T}$ in the Pick's-type formula for 3D seems like a pain, there seem to be four pieces of information if you consider the interior and boundary lattice points separately for the original polyhedron and the doubled polyhedron. Is there a Pick's-type formula in three dimensions that uses just three out of these four quantities? What about in four dimensions, where we presumably need four quantities: is there a Pick's-type formula that uses just these four quantities?