Functions and their Rates of Change II

Preliminary Version

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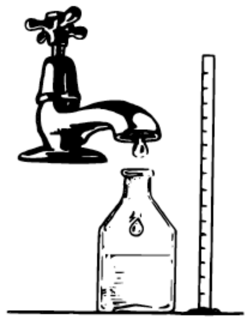
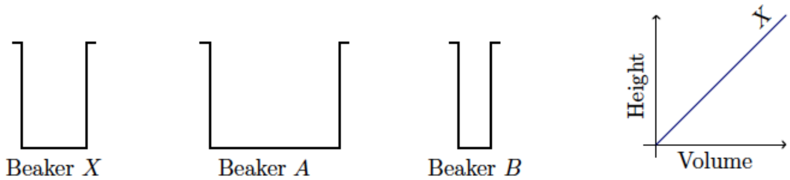
1 Bottle Calibration



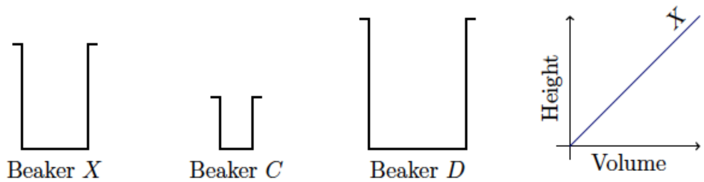
Filling Bottles

In order to calibrate a bottle so that it may be used to measure liquids, it is necessary to know how the height of the liquid depends on the volume in the bottle.

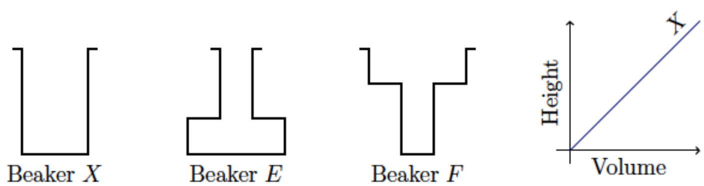
1. The graph below shows how the height of the liquid in beaker X varies as water is steadily dripped into it. On the same diagram, show the height-volume relationship for beakers A and B.



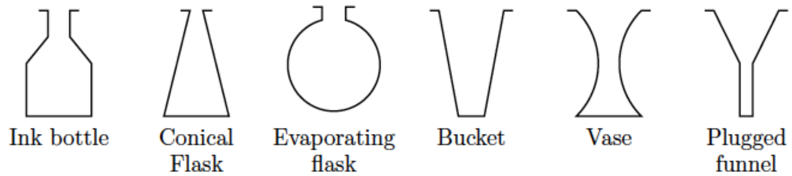
2. Sketch two more for C and D:

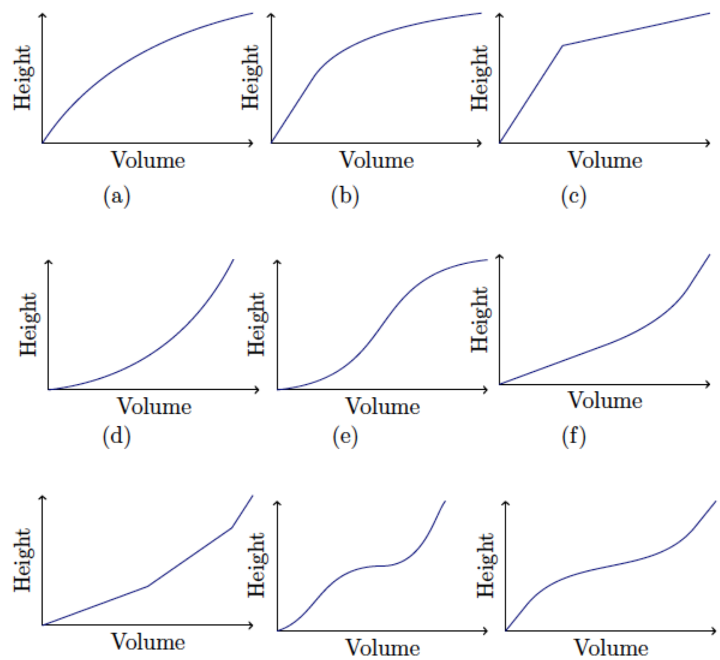


3. Sketch two more for E and F:



4. Here are six bottles and nine graphs. Choose the correct graph for each bottle. Explain your reasoning clearly. For the remaining three graphs, sketch what the bottles should look like.





(a) (b) (c) (d) (e) (f) (g) (h) (i)

2 All About Functions

1. Consider the time that it would take to pick all the apples at one particular orchard as a function of the number of workers picking the apples.

   (a) Sketch a graph of the time it takes to harvest the crop of apples as a function of the number of people picking apples. Label your axes.

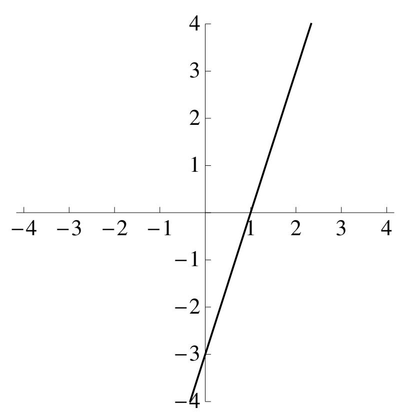
(b) Is the graph you drew a straight line? Why or why not?

(c) Is the graph you drew continuous or discontinuous? Explain.

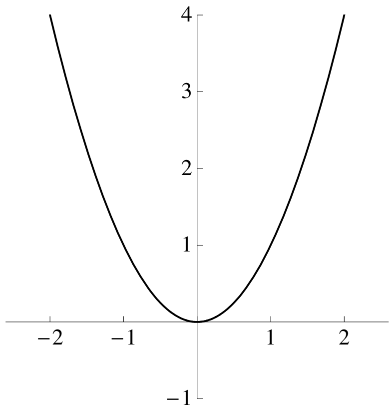
(d) Does the graph you drew intersect the horizontal axis? The vertical axis? If so, where? If not, why not?

2. In each part, think of a function $f(x) =$ _______ whose graph looks like the given picture.

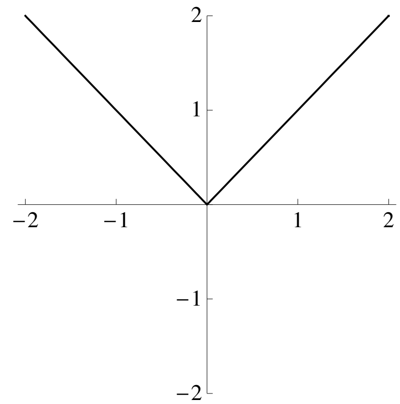
(a)



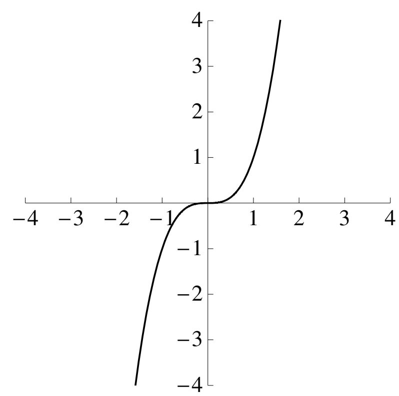
(b)



(c)



(d)



3. Sketch the graph of each of the following functions.

(a) $f(x) = \dfrac{1}{x}$.

(b) $f(x) = -2x + 1$.

(c) $f(x) = \dfrac{1}{x^2}$.

(d) $f(x) = \sqrt{x}$.

4. Come up with definitions of even and odd:

   - A function $f(x)$ is even if ____.
   - A function $f(x)$ is odd if ____.

   Use the definitions to decide whether each of the following is even, odd, or neither.

   (a) $f(x) = -3x^4 + 6x^2 + 1$.

   (b) $g(x) = x^3 + x^2 + 1$.

3 Modeling with Functions

1. A Harvard professor decides to exercise the traditional "grazing rights" that come with the job, namely the right to graze animals in Harvard Yard[(1)]. He wants to have some room for a cow to roam in without escaping, so he decides to build a rectangular fenced-in pen, using 28 feet of wire fencing. He is convinced that it does not matter what dimensions his fenced-in-pen is, the area will be constant. Help the professor see how area changes as length changes.

   (a)

   (b)

   (c)

   (d)

[(1)] See news.harvard.edu/gazette/story/2009/09/grazing-rights/.

2. A cistern for storing water is in the shape of a right circular cone with radius 10 feet and height 40 feet. The volume of a cone with radius R and height H is $\frac{1}{3}\pi R^2 H$.

   (a) Suppose the cistern is filled part-way with water. Express the height of the water as a function of the radius of the surface of the water.

   (b) Express the volume of water in the cistern as a function of the radius of the surface area of the water. What are the domain and range of this function?

   (c) Express the volume of water in the cistern as a function of the height of water in the cistern. What are the domain and range of this function?

   (d) Now, suppose you want to find the calibration function C for the cistern, which takes as input the volume of water in the cistern and produces as output the height of water in the cistern. Sketch a graph of $C(v)$. What are the domain and range of $C(v)$?

   (e) Can you find a formula for $C(v)$?

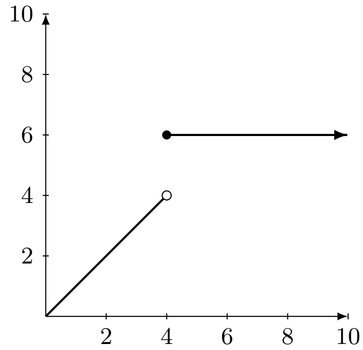
3. The cost of sending a first-class large envelope through the U.S. Postal Service depends on the weight of the letter. As of September 2016, it costs \$0.94 for the first 3 ounces and \$0.21 for each additional ounce up to a maximum of 8 ounces.

   (a) Draw a graph of the cost of package as a function of its weight.

   (b) What are the domain and range of the cost function?

   (c) Is the function continuous?

4. Write a formula for a function $f(x)$ whose graph looks like the following picture.



5. Let $d(x)$ be the distance between x and 5 on the number line.

   (a) What is the domain of $d(x)$? What is the range of $d(x)$?

   (b) For what values of x is $d(x) = 0$?

   (c) For what values of x is $d(x) = 1$?

   (d) For what values of x is $d(x) > 1$?

   (e) Graph $d(x)$.

   (f) Write a formula for $d(x)$.

4 More on Functions; Average Rates of Change

1. Imagine a road trip from Route 66 from Flagstaff, Arizona, through New Mexico and Texas and into Oklahoma. Two functions could help describe this trip:

   i. The function f gives the number of miles traveled t hours into the trip, where $t = 0$ denotes the beginning of the trip.

   ii. The function g gives the car's speed t hours into the trip, where $t = 0$ denotes the beginning of the trip.

   Each of these functions is closely related to one of the instruments on the car's dashboard. At time t, $f(t)$ corresponds to the change in the ______________ reading since the friends set off from Flagstaff. At that moment, $g(t)$ corresponds to the ______________ reading.

   Suppose they pass a sign that reads "entering Gallup, New Mexico," T hours into the trip. Write the following expressions using functional notation wherever appropriate.

   (a) The car's speed two hours before entering Gallup.

   (b) The car's average speed in the first 3 hours of the trip.

   (c) The car's average speed in the second three hours of the trip.

   (d) The car's average speed in the half an hour after getting to Gallup, New Mexico.

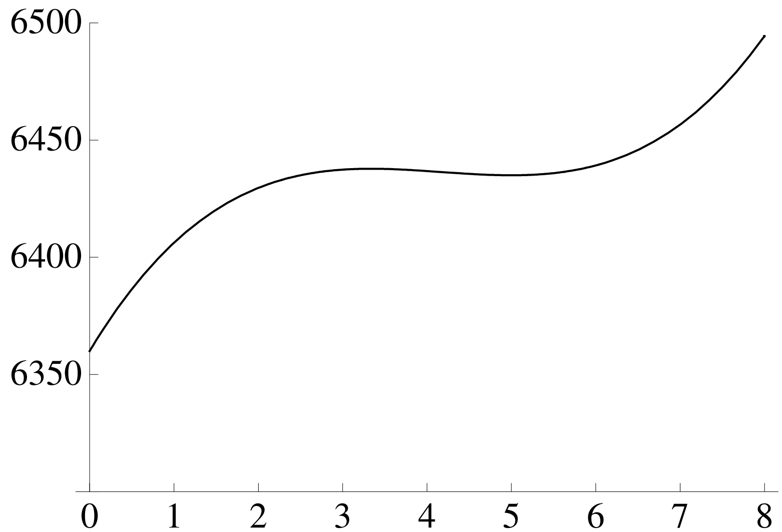
2. Consider again the functions from the previous problem.

   Interpret the following quantities in words.

   (a) $\dfrac{f(T+5) - f(T)}{5}$

   (b) $\dfrac{g(T+0.1) - g(T)}{0.1}$

3. Based on U.S. Census data[2], the population of Massachusetts between 2000 and 2008 can be modeled by the function $P(t) = 6360 + 60t - 15t^2 + 1.2t^3$, where $P(t)$ gives the number of thousands of people in Massachusetts t years after July 1, 2000.



(a) According to this model, what was the population of Massachusetts on July 1, 2000?

(b) What was the change in the population population between July 1, 2000 and July 1, 2001? Use functional notation.

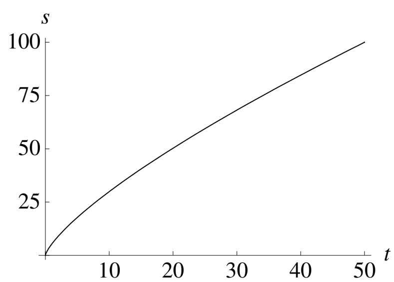
(c) What was the average rate of change of the population of Massachusetts between July 1, 2000 and July 1, 2004? What are the units of your answer?

(d) What was the average rate of change of the population of Massachusetts between July 1, 2004 and July 1, 2008?

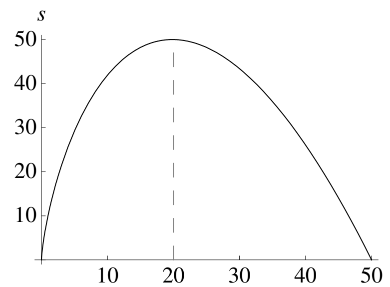
[2] http://factfinder.census.gov/servlet/GCTTable?_bm=y&-geo_id=01000US&-ds_name=PEP_2008_EST&-_lang=en&-redoLog=false&-mt_name=PEP_2008_EST_GCTT1_US40&-format=US-40&-CONTEXT=gct

4. A swimmer is swimming a 100 m long race (one lap in a 50 m long pool). Let $s(t)$ be the swimmer's distance from the starting position t seconds after the start of the race.

   (a) Which of the following is a more reasonable graph for $s(t)$? Why?



(A)



(B)

   (b) According to the graph you chose, how long did it take the swimmer to finish the race?

   (c) What was the swimmer's average velocity over the first 20 seconds of the race?

   (d) What was the swimmer's average velocity over the last 50 m of the race?

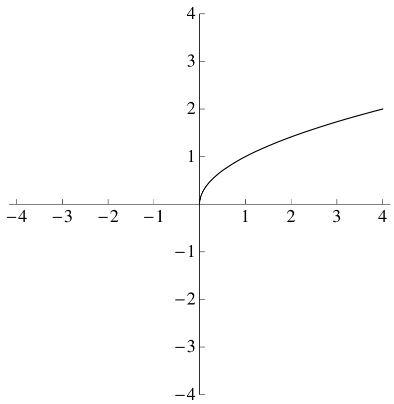
   (e) What is the difference between velocity and speed?

5. Suppose $f(x)$ is a linear function defined on $(-\infty, \infty)$. How does the average rate of change of f on $[0, 5]$ compare to the average rate of change of f on $[0, 500]$? What about the average rate of change of f on $[-50, 2]$?

5 Altering Functions

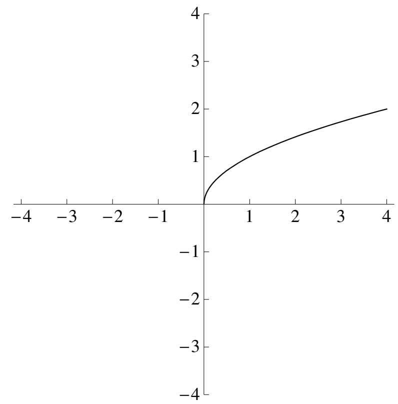
1. In each part, the graph of $f(x) = \sqrt{x}$ is shown. Write a formula for the specified function, and sketch its graph. Describe the relationship between the graph of the new function and the graph of $f(x)$.

   (a) $f(x-2) =$ ______________

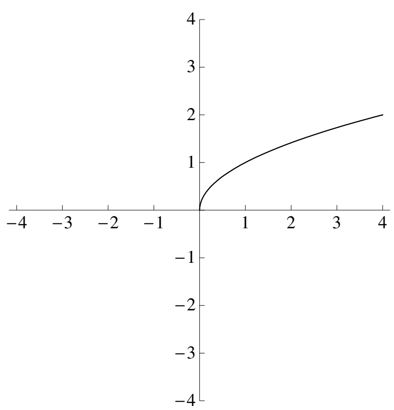
   Description:

   (b) $f(x) - 2 =$ ______________

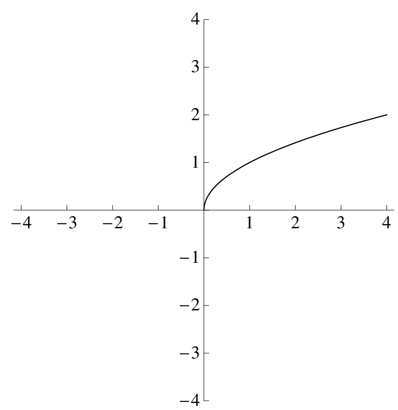
   Description:

   (c) $f(-x) =$ ______________

   Description:

   (d) $-f(x) =$ ______________

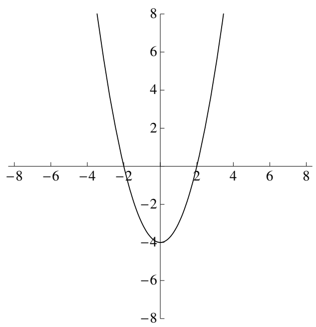
   

   Description:

2. (a) Sketch the graph of $f(x) = x^2 - 4$. Where are its x- and y-intercepts?

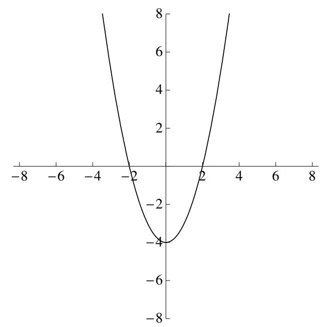
Let $f(x) = x^2 - 4$. In each part, write a formula for the specified function, and graph it together with $f(x)$. Describe in words how the graph of the new function is related to the graph of $f(x)$.

(b) $f(2x) =$ ____________________



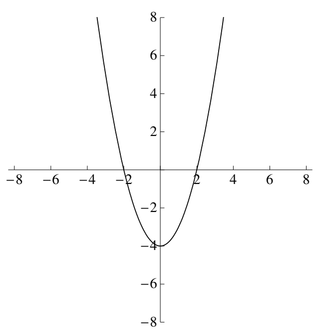
Description:

(c) $2f(x) =$ ____________________



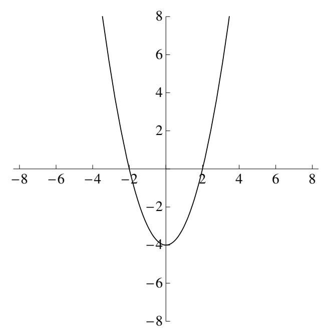
Description:

(d) $-2f(x) =$ ____________________



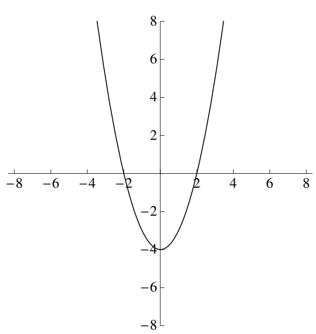
Description:

(e) $f\left(\dfrac{x}{2}\right) =$ ____________________



Description:

(f) $\dfrac{f(x)}{2} =$ ____________________



Description:

3. Sketch the following, and label any asymptotes. For (b) - (e), explain how the graph is related to the graph of $\frac{1}{x^2}$.

(a) $y = \dfrac{1}{x^2}$.

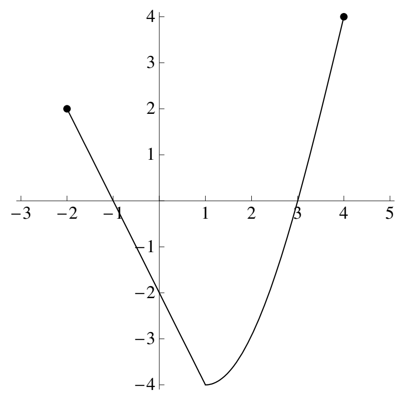
(b) $y = \dfrac{3}{x^2} - 1$.

(c) $y = 3\left(\dfrac{1}{x^2} - 1\right)$.

(d) $y = \dfrac{3}{(x+2)^2} - 1$.

(e) $y = -\dfrac{3}{(x+2)^2} - 1$.

4. Here is the graph of a function f.



(a) What is the domain of f?

(b) What is the range of f?

(c) Sketch graphs of the following functions. Where are the two endpoints of the graph?

i. $f(-2x)$

ii. $-2f(x)$

iii. $-2f(x+1)+3$

iv. $-2f(2x)+3$

6 Linear Functions Local Linearity

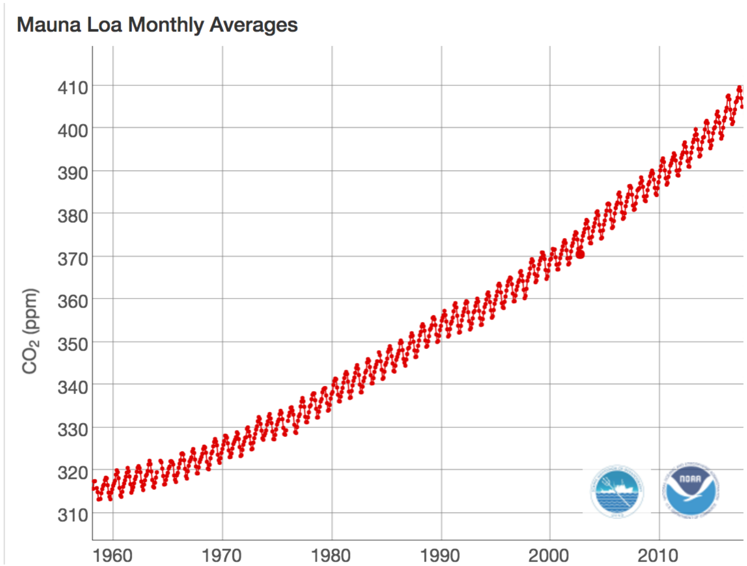
1. Here is some sunset/sunrise information for Cambridge, MA in 2018:

	19 Sept	20 Sept	21 Sep	22 Sept
Sunrise	6:29 am	6:30 am	6:31 am	6:32 am
Sunset	6:46 pm	6:44 pm	6:43 pm	6:41 pm

   (a) Estimate when the sun will rise on September 30th.

   (b) Would you feel comfortable using the same technique to estimate the sunrise and sunset in November? In March? Why or why not?

2. At the Mauna Loa observatory in Hawaii, the amount of carbon dioxide in the atmosphere has been measured every month since 1958. Here is a graph of the data.



Generally, the carbon dioxide concentration each year is highest in May and lowest in October.

(a) A scientist is looking at some recent measurements, from August 2017 (405 ppm) and May 2017 (410 ppm). If she uses local linearity to predict the CO_2 concentration in September 2017, do you expect her to get a reasonable estimate? What if she uses the same method to predict the CO_2 concentration in May 2018?

(b) Find the equation of a line that the scientist could use to predict the CO_2 levels in September 2017.

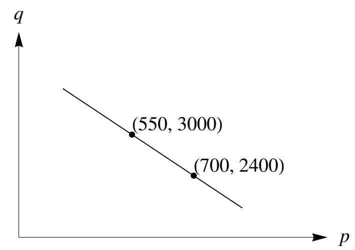
3. Find an equation for each of the following lines.

   (a) The line with slope $-\frac{1}{5}$ passing through the point $(-1, 5)$.

   (b) The line passing through the points $(0, p)$ and $(q, 0)$.

   (c) The line passing through the point $(\sqrt{3}, 1)$ and parallel to the line $\sqrt{3}x - 2y = 5$.

4. Economists use demand curves to express the relationship between the price of an item and the number of items demanded by consumers. Below is the demand curve for a certain good. p is the price per item measured in dollars, and q is the quantity of the good demanded (i.e., the number of items demanded).



(a) Based on the graph above, express the demand q as a function of the price p.

(b) In words, explain the meaning of the slope of the line.

5. Molly has just started a part time job that pay \$10 per hour for the first 6 hours that she works each week, and \$12 per hour for each hour thereafter, up to a total of 15 hours per week.

(a) Find an expression for the function $P(t)$ whose input is the number of hours that Molly works in a week, and whose output is her income for that week.

(b) Sketch the graph of $P(t)$.

(c) Graph $P'(t)$, where $P'(t)$ is the slope function, which gives the slope at every point on the graph of P. (Are there places where the slope is not defined?)

(d) What is the practical meaning of $P'(t)$? Include units in your answer.

7 Modeling and Interpreting Slope

Often we can use a line to help us approximate the values of a function on a curve. We use a line because the line is sometimes much easier to work with than the curve itself. Here is an example.

1. Sketch the graph of $f(x) = \sqrt{x}$. Find the equation of the secant line through the points whose $x-$ coordinates are 4 and 9.

   (a) What is the average rate of change of the function $f(x) = \sqrt{x}$ on an interval $[4, 9]$? How would you interpret this on the graph of $f(x)$?

   (b) Use the secant line you found to estimate $\sqrt{5}$. Is your estimate too high or too low? Explain.

   (c) Use the secant line you found to estimate $\sqrt{3}$. Is your estimate too high or too low? Explain.

(d) Find the slope of the secant line through the points where $x = 4$ and $x = 4 + h$. Your answer will be in terms of h.

2. **Thomas Wolfe's Royalties for** *The Story of a Novel* [(3)]

In lieu of a straight royalty percentage on sales of his book, The Story of a Novel, Thomas Wolfe agreed to accept a sliding scale of 10% on the first 3000 copies sold, 20% on the next 1500 copies, and 30% on all copies after that. The book was priced at \$2.00. [(4)]

(a) Write a function $R(x)$ that gives Wolfe's royalties as a function of x, the number of books sold.

(b) Graph $R(x)$.

(c) Solve $R(x) = 900$. What does this equation mean?

[(3)] Thomas Wolfe was an author born in Asheville, North Carolina, in 1900. His first novel, *Look Homeward Angel*, was a thinly veiled fiction that gave such scathingly accurate portrayals of well-known townspeople that despite the novels enthusiastic reception nation- wide, Wolfe felt he received a cool welcome in his hometown. He proceeded to write $\hat{Y}$ou Can't Go Home Again. The problem you are asked to consider involves one of Wolfe's lesser-known novels.

[(4)] (Elizabeth Nowell, Thomas Wolfe: A Biography, Doubleday, Garden City, NY, 1960. Reported percents have been modified for computational ease.)

(d) Graph $R'(x)$, where $R'(x)$ is the slope function, a function giving the slope of the graph of R.

(e) What is the practical meaning of $R'(x)$? Include units in your answer.

8 Definition of the Derivative

1. A rocket is launched straight up into the air. The function $f(t) = t^2$ describes the rocket's height in meters t seconds after liftoff. Suppose you'd like to know the rocket's velocity 5 seconds after liftoff.

   (a) Sketch the graph of the position function. (Would this be a good model for the position function in the latter portion of the journey?)

   (b) How can the instantaneous velocity of the rocket be approximated at $t = 5$ seconds? Write an expression that approximates the instantaneous velocity.

   (c) How could we find the exact instantaneous velocity of the rocket at $t = 5$ seconds? How would you visualize this instantaneous velocity graphically?

Definition and interpretation of the derivative. If we have a function $f(x)$ and a number a, we define the derivative of f at a, written $f'(a)$, to be

$$f'(a) = \boxed{} \quad \text{or} \quad \boxed{}$$

We often read $f'(a)$ as "f prime of a."

- We interpret $f'(a)$ as the instantaneous rate of change of f at $x = a$.
- Graphically, we visualize $f'(a)$ as the slope of the line tangent to the graph of f at $x = a$.

2. Let $f(x) = \dfrac{1}{4x}$.

   (a) Sketch the graph of $f(x)$ and the line tangent to f at $x = \frac{1}{4}$. Is the slope of the line positive or negative?

   (b) Calculate $f'\left(\frac{1}{4}\right)$ exactly using the definition of the derivative. Is this consistent with your answer to (a)?

   (c) What is the equation of the line tangent to f at $x = \frac{1}{4}$?

   (d) Can you think of another way to check whether your answer to (b) is reasonable?

   (e) Would you expect $f'(1)$ to be positive or negative? How would that compare with $f'\left(\frac{1}{4}\right)$?

   (f) Would you expect $f'\left(\frac{-1}{4}\right)$ to be positive or negative? How would that compare with $f'\left(\frac{1}{4}\right)$?

3. Let $A(r)$ be the area of a circle of radius r cm.

   (a) Find the derivative $A'(5)$. What are its units? What does it represent?

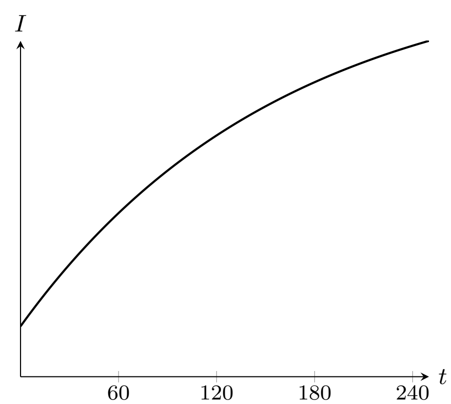
   (b) Can you draw a picture involving a circle that explains why your answer makes sense?

4. Peter is roasting a 14 lb turkey as a test run for Thanksgiving dinner. He begins at noon. At 1:00pm, he checks on the temperature and discovers that it has an internal reading of 35.6°C,[5] and it's rising at an instantaneous rate of 0.25°C per minute.

   (a) Approximate the temperature of the turkey at 1:06pm.

   (b) Let $I(t)$ be the turkey's internal temperature (in °C) t minutes after noon. Use functional notation to express what you were told about the turkey.

   (c) Here is a graph of $I(t)$. Use a sketch to explain the approximation you made in (a).



   (d) Based on your sketch, was your approximation too high or too low?

   (e) Would you be comfortable using the same method to predict the turkey's temperature at 3 pm? Explain.

[5] According to the USDA, a turkey should be roasted to an internal temperature of 73°C.

9 The Derivative Function

1. Let $f(t) = t^2$.

   (a) We can define a new function $f'(t)$ that gives slope of the line tangent to the graph of f at the point $(t,f(t))$. What makes this a function?

   (b) For what values of t will $f'(t)$ be positive?

   (c) For what values of t will $f'(t)$ be negative?

   (d) Before we calculate $f'(t)$, describe some qualities

   (e) Write a formula for $f'(t)$.

2. Gromit has been growing a giant squash for Tottington Hall's annual Giant Vegetable Competition. He has carefully tracked his squash's length and weight. Let $w(\ell)$ be the squash's mass in kg when its length is ℓ cm.

   (a) What are the units of $w'(\ell)$?

   (b) Another notation for $w'(\ell)$ is $\frac{dw}{d\ell}$. Do you expect $\frac{dw}{d\ell}$ to be positive or negative? Why?

   (c) Suppose $w'(70) = 8$. (This statement can also be written as $\left.\frac{dw}{d\ell}\right|_{\ell=70} = 8$.) Which is the following is the most reasonable conclusion?

      A. It takes 70 days for the squash to grow to be 8 kg.

      B. When the squash is 70 cm long, it weighs 8 kg.

      C. When the squash is 70 cm longer than its current length, it will weigh 8 kg more than it currently does.

      D. When the squash grows from 70 cm to 71 cm, it will gain about 8 kg in mass.

   (d) Interpret the statement $w'(105) = 10$ in words.

3. (a) Sketch the graph of $|x-1|$.

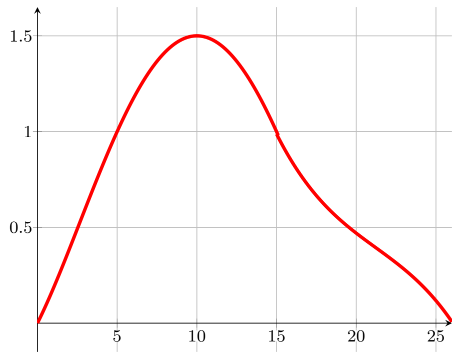
   (b) Sketch the graph of $\dfrac{d}{dx}|x-1|$. (The notation $\dfrac{d}{dx}|x-1|$ is shorthand for "the derivative of $|x-1|$".) There's one particularly interesting value of x here; what's going on at that value of x?

   (c) Generalize. Looking at the graph of $g(x)$, what is one way we could know that $g'(x)$ does not exist.

4. Let $f(x) = \left| \dfrac{x^4 - 9x^2 + 20}{x^2 - 4} \right|$. Sketch the graph of f, and then sketch the graph of f'.

10 Functions and their Derivatives

1. **Velocity and speed, average and instantaneous.** The #68 bus runs between Harvard Square and Kendall Square. The first bus leaves Harvard Square at 6:35 am. Let $d(t)$ be the bus's distance (in miles) from Harvard Square t minutes after 6:35 am. Here is the graph of $d(t)$:



(a) How many miles long is the bus route?

(b) At what time does the bus reach Kendall Square? At what time does it return to Harvard Square?

(c) Sketch a graph showing the bus's velocity t minutes after 6:35 am.

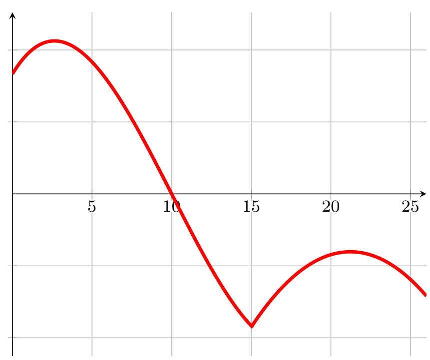
(d) Sketch a graph showing the bus's speed t minutes after 6:35 am.

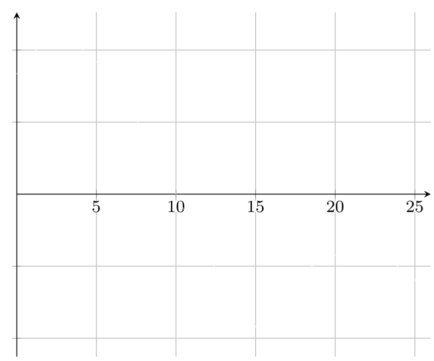
(e) What is the bus's average velocity over this round trip?

(f) What is the bus's average speed over this round trip?

(g) At roughly what times does the bus have positive acceleration? At roughly what times does it have negative acceleration?

(h) Below is a graph that represents $v(t)$, the velocity of the bus t minutes after 6:35 am. Sketch a graph for the buses acceleration t minutes after 6:35 am.

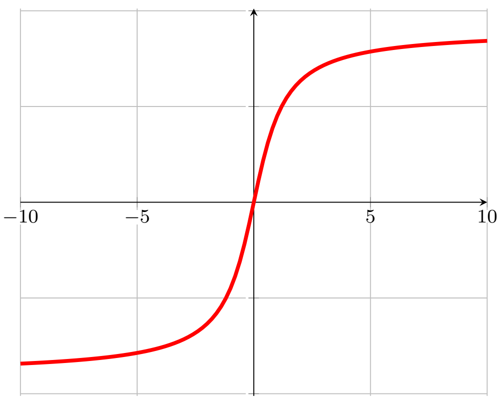




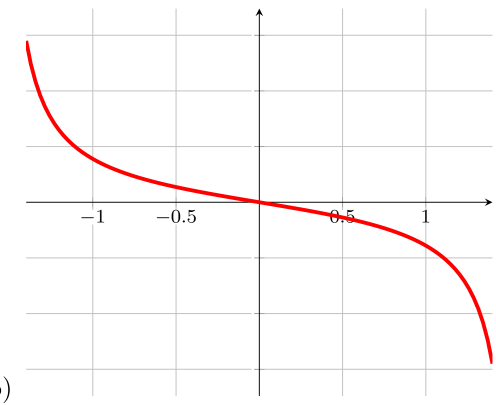
(i) Draw some connections between the graph of the acceleration and the velocity.

2. In each part, you are given the graph of a function $f(x)$. Sketch a graph of $f'(x)$.

(a)

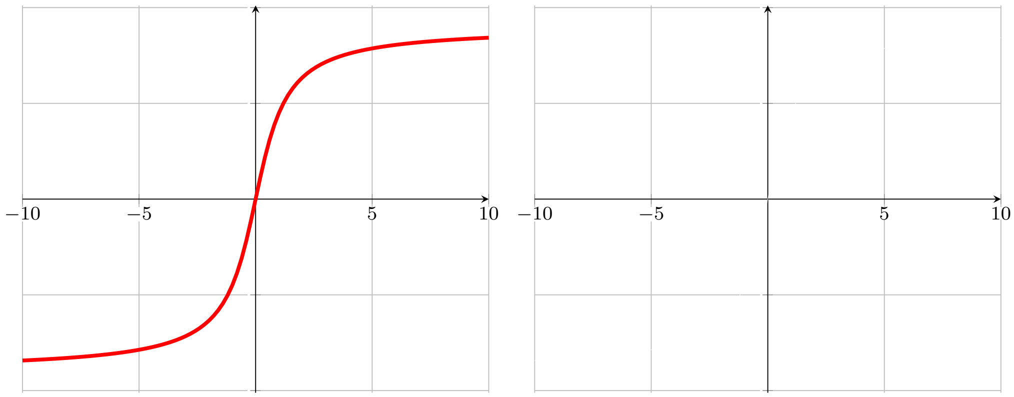


(b)



(c) What strategies were particularly helpful?

3. Below is a graph of $g'(x)$. Draw a graph for $g(x)$.



4. **Recap:** What does the graph of f tell you about f'? What does the graph of f' tell you about f?

   Fill in the following charts. If a given piece of information about one function tells you nothing about the other function, put an NA in the corresponding box.

f	f'
$+$	
$-$	
$\nearrow$	
$\searrow$	
$\cup$	
$\cap$	

f'	f
$+$	
$-$	
$\nearrow$	
$\searrow$	
$\cup$	
$\cap$	

11 Limits

1. If $f(x) = x^2$, calculate $f'(7)$ using the definition of the derivative.

Limits. You can think of the notation $\lim_{x \to a} f(x)$ as shorthand for:

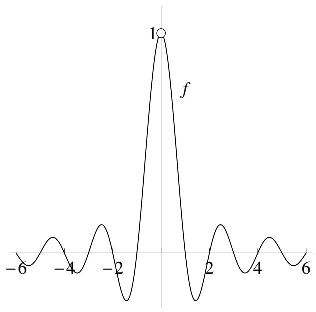
"What number (if any) does $f(x)$ approach as x gets really close to a, without actually being equal to a?"

This number is called the **limit of $f(x)$ as x approaches a**.

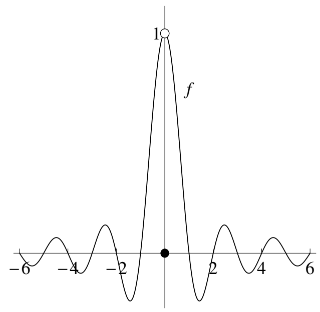
More precisely, saying $\lim_{x \to a} f(x) = L$ means that we can make the values of $f(x)$ as close as we like to L by making x sufficiently close to (but not equal to) a.

2. In each of the following examples, what is $\lim_{x \to 0} f(x)$? What is $f(0)$?

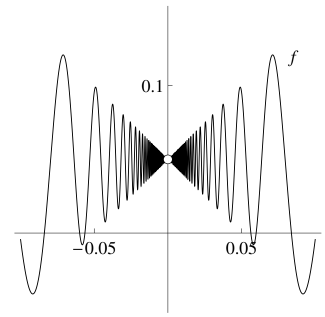
(a)



(b)



(c)



If a is a number, x can get really close to a from two sides, the left and the right. So, we also look at <u>one-sided limits</u>:

- The <u>limit of $f(x)$ as x approaches a from the *left*</u>, $\lim_{x \to a^-} f(x)$, is the number that $f(x)$ approaches as x gets really close to a, while remaining slightly *less* than a.
- The <u>limit of $f(x)$ as x approaches a from the *right*</u>, $\lim_{x \to a^+} f(x)$, is the number that $f(x)$ approaches as x gets really close to a, while remaining slightly *greater* than a.

3. Sketch an example of a function f for which $\lim_{w\to 2^-} f(w) = 1$ and $\lim_{w\to 2^+} f(w) = 3$. Can you sketch an example which also has the property that $\lim_{w\to 2} f(w)$ exists?

4. Use the definition of the derivative to find $f'(0)$ for $f(x) = |x|$.

12 Working with Limits

1. Sketch the graph of $f(x) = \frac{1}{x}$, and use it to find the following limits.

   (a) $\lim_{x \to 0^+} \frac{1}{x}$

   (b) $\lim_{x \to 0^-} \frac{1}{x}$

   (c) $\lim_{x \to 0} \frac{1}{x}$

   (d) $\lim_{x \to \infty} \frac{1}{x}$

   (e) $\lim_{x \to -\infty} \frac{1}{x}$

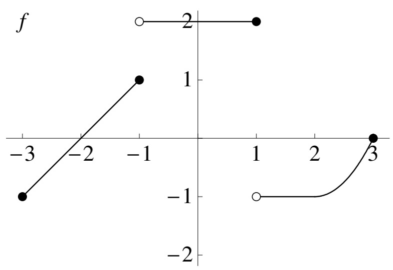
2. Calculate the following limits:

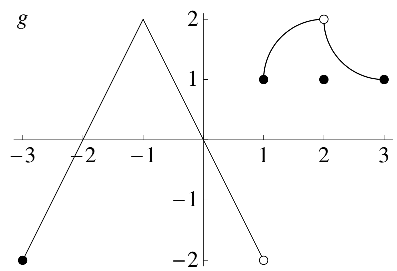
   (a) $\displaystyle\lim_{x \to \infty} \frac{x^2 + x + 1}{x + 1}$

   (b) $\displaystyle\lim_{x \to \infty} \frac{x^2 + x + 1}{x^2 + 1}$

   (c) $\displaystyle\lim_{x \to \infty} \frac{x^2 + x + 1}{x^3 + 1}$

3. The graphs of f and g are given below.





Evaluate each of the following quantities.

(a) $\lim_{x\to 2} g(x)$

(b) $g(2)$

(c) $\lim_{x\to 2} \dfrac{f(x)}{g(x)}$

(d) $\lim_{x\to 1} \dfrac{f(x)}{g(x)}$

(e) $\lim_{x\to -1} \dfrac{f(x)}{g(x)}$

(f) $\lim_{x\to -2} \dfrac{f(x)}{g(x)}$

4. When calculating limits what should your strategies be?

5. Sketch a graph of the relevant function, and use it to find the specified limit.

(a) $\displaystyle \lim_{x\to 2^+}\left(3-\frac{4}{x-2}\right)$ (b) $\displaystyle \lim_{x\to 2^-}\left(3-\frac{4}{x-2}\right)$ (c) $\displaystyle \lim_{x\to \infty}\left(3-\frac{4}{x-2}\right)$

13 Limit Laws and Continuity

1. **Warm-up (from PS13).** *When is* $\lim_{x \to a} f(x) = f(a)$*? That is, when can you just plug in to evaluate a limit?*

   (a) Sketch some functions $f(x)$ for which $\lim_{x \to 3} f(x) \neq f(3)$. Try to draw a few examples that look really different!

   (b) Sketch some functions $f(x)$ for which $\lim_{x \to 3} f(x) = f(3)$. Try to draw a few examples that look really different!

2. In the past, we calculated $f'(7)$ for $f(x) = x^2$:

$$\begin{aligned} f'(7) &= \lim_{h \to 0} \frac{f(7+h) - f(7)}{h} = \lim_{h \to 0} \frac{(7+h)^2 - 7^2}{h} && \text{by the definition of the derivative} \\ &= \lim_{h \to 0} \frac{h^2 + 14h + \cancel{49} - \cancel{49}}{h} = \lim_{h \to 0} \frac{h(h+14)}{h} && \text{by algebra} \end{aligned}$$

   If we plug in 0 for h, what is the result?

   Does $h = 0$ as we take this limit? For all nonzero values of h, what is $\frac{h(h+14)}{h}$?

$$\begin{aligned} &= \lim_{h \to 0} \underline{\hspace{4cm}} && \text{for values of } h \neq 0 \\ &= \underline{\hspace{4cm}} && \underline{\hspace{9cm}} \end{aligned}$$

Whether limits exist and agree is directly connected to whether their graphs can be sketched without lifting our pencil. As a result, we can update our definition of continuity.

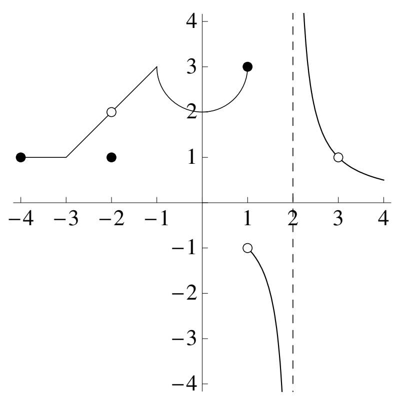
A function is considered **continuous** at some point $x = a$ if

$$\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = f(a)$$

.

That is, a function is said to be continuous at some point if the right and left hand limits of the function as we approach the point are the same, and the value of those limits is the function's value at that point.

3. Here is the graph of a function $f(x)$.



(a) At what values of x in $(-4, 4)$ is $f(x)$ *not* continuous? In other words, at what values of x does $f(x)$ have a discontinuity?

(b) At what values of x in $(-4, 4)$ is $f(x)$ *not* differentiable?

Now that we're more comfortable with limits, we can get more technical about what it means for a function to be differentiable.

We consider a function to be **differentiable** at some point $x = a$ if the two-sided limit $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.

(c) Decide whether each of the following statements is true or false. If it is false, give an example that shows that it's false (such an example is known as a *counterexample*).

i. If $f(x)$ is continuous at $x = a$, then $f(x)$ is differentiable at $x = a$.

ii. If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

4. In each part, can you sketch a continuous function $f(x)$ defined on $[-4, 3]$ which has all three of the given properties, or is it impossible to do so?

   If it is impossible, would it become possible if f did not have to be continuous?

   (a)
   - $f(-4) = 2$
   - $f(3) = 5$
   - f has no zeros in $[-4, 3]$

   (b)
   - $f(-4) = 2$
   - $f(3) = 5$
   - f has a zero in $[-4, 3]$

   (c)
   - $f(-4) = 2$
   - $f(3) = -1$
   - f has no zeros in $[-4, 3]$

   (d)
   - $f(-4) = 2$
   - $f(3) = -1$
   - f does not attain the value 1 on $[-4, 3]$

Intermediate Value Theorem. If f is continuous on the closed interval $[a, b]$ and $f(a) = A$, $f(b) = B$, then somewhere in the interval $[a, b]$, f attains every value between A and B.

5. Is the following argument correct or incorrect? Why?

   Imagine that tonight Harvard Football Team beats Holly Cross 29-13. Since Harvard will 0 points at the beginning of the game and 29 at the end, the Intermediate Value Theorem says that Harvard must will have exactly 25 points at some moment in the game.

6. Let f be the function defined by

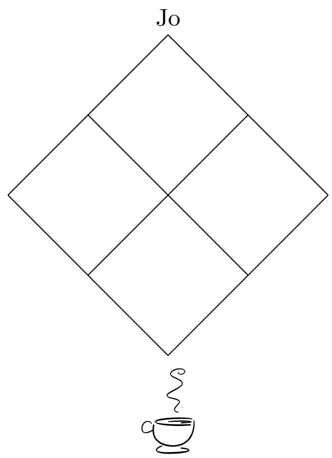
$$f(x) = \begin{cases} -x^2 + 1 & \text{for } x \leq 1 \\ ax + b & \text{for } x > 1 \end{cases}$$

   (a) For what values of a and b will f be continuous at $x = 1$?

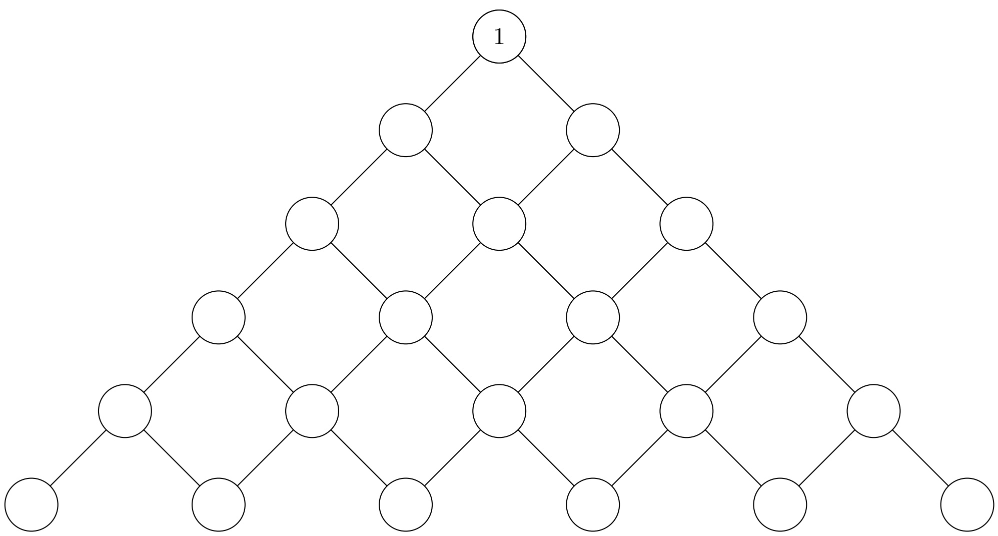
   (b) Are there any values of a and b for which f will be differentiable at $x = 1$?

14 Patterns in Paths

1. Jo desperately needs some coffee, and it's a short walk to the nearest coffee shop. If she must walk along the roads shown, how many different paths can she take to get to the coffee shop as quickly as possible?



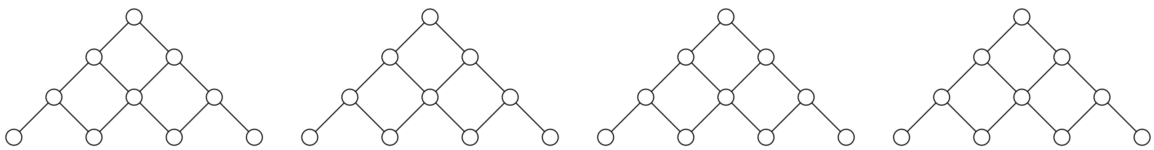
2. In the picture below, how many ways can Jo get to each intersection if she starts from the top point?



What patterns do you notice? Can you explain them?

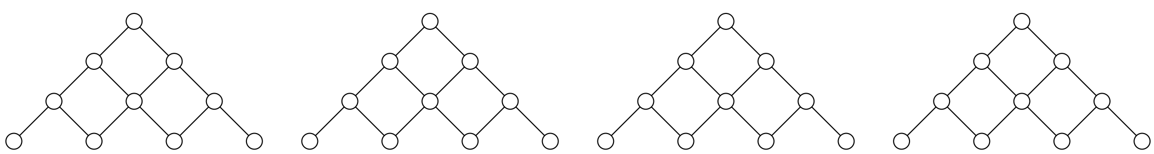
3. (a) Ron needs to expand $(L + R)^2$, and his first guess is that the answer is $L^2 + R^2$. Help Ron understand why this is incorrect, and explain the correct answer to him.

   (b) Can you visualize each term in the expansion as a path?



4. (a) Use the distributive property to compute $(L + R)^3$.

   (b) Can you visualize each term in the expansion as a path?



5. Use Pascal's Triangle to calculate $(L + R)^4$.

6. (a) Calculate $(x + y)^6$.

   (b) What is $(x - z)^6$?

15 Differentiation Rules

You have found the derivatives of many functions – fill in the table below.

function	derivative
3	
x	
x^2	
$x^{-1} = \frac{1}{x}$	
$x^{-2} = \frac{1}{x^2}$	

1. Use the definition of the derivative to find $\dfrac{d}{dx}(x^4)$. Hint: you might want to use Pascal's triangle.

2. Let $f(x) = x^n$. Use your results to make a conjecture about $f'(x)$.

3. Verify your conjecture by using the limit definition of the derivative to find $\dfrac{d}{dx}(x^n)$.

4. What do you think the derivative of $f(x) + g(x)$ is? What do you think the derivative of $cf(x)$ is?

   (a) How can we make sense of your two conjectures?

   (b) Use the limit definition to support your conjecture.

5. Let $f(x) = x^2 + x + 1$ and $g(x) = 3x^2 + 1$.

   (a) Calculate $f'(x)$.

   (b) Calculate $g'(x)$.

   (c) What is the derivative of $f(x)g(x)$?

6. At noon one day, Jack plants a magical bean. To his delight, it grows quickly into a giant beanstalk. Let $f(t)$ be the beanstalk's height in feet t hours after noon.

   (a) If $f'(3) = 7$, roughly how much does the beanstalk grow between 3:00 pm and 3:06 pm?

   (b) More generally, if h is a fairly small positive number, write an approximation for the amount that the beanstalk grows between times t and $t + h$.

7. Use the limit definition to calculate the derivative of $f(x)g(x)$.

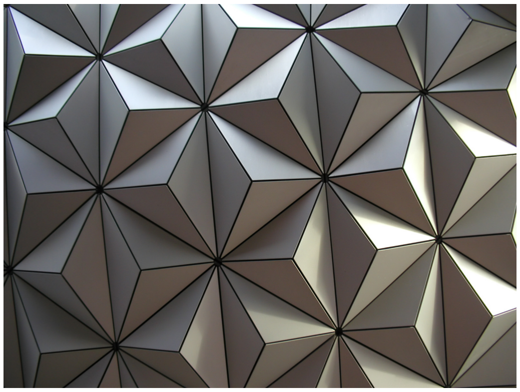
Differentiation Rules to Remember!

Civilization advances by extending the number of important operations which we can perform without thinking about them. – Alfred Whitehead, mathematician & philosopher

- **Power Rule:**
- **Sum Rule:** $\dfrac{d}{dx}[f(x) + g(x)] =$
- **Constant Multiple Rule:** If c is a constant, then $\dfrac{d}{dx}[cf(x)] =$
- **Product Rule:** $\dfrac{d}{dx}[f(x) \cdot g(x)] =$
- **Quotient Rule (this is on your homework!):** $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)g(x)}$

16 Linear Approximation

1. Walt is designing a 18-story geodesic sphere that takes guests on a time machine-themed experience. Walt knows the geodesic sphere will require about 10 thousand equilateral triangles. If he wants the side length of each equilateral triangle to be 1 foot, what is the surface area of Walt's imaginative idea?



2. How would you approximate $\sqrt{3}$?

   (a)

   (b)

3. *In this problem, you'll use the Product Rule to find the derivative of* $\sqrt{x}$. We know that $\sqrt{x} \cdot \sqrt{x} = x$. If we let $f(x) = \sqrt{x}$, we can rewrite this fact as: $f(x)f(x) = x$.

   (a) Differentiate both sides of the equation $f(x)f(x) = x$ to write a new equation involving $f(x)$ and $f'(x)$. (You'll need to use the Product Rule to differentiate the left side.)

   (b) Solve the equation you wrote in (a) for $f'(x)$.

   (c) Why does the power rule we learned in the last class not just quickly apply here?

4. How could you approximate $\sqrt[3]{9}$?

   (a) Method 1:

   (b) Method 2:

17 Constant Percent Change

1. On his birthday last year, Kyle invested \$1,000 in a small company. Good move Kyle. This year, he discovered that his investment had grown by 10% and is now worth \$1,100.

   (a) Investment plan version 1:

      i. Suppose that the investment grows at a constant rate. Make a table that shows the value of his investment over the course of the next 5 years.

      ii. Write down a function $f(x)$ that gives the balance in his bank account x years after his initial deposit.

      iii. By what percentage will the investment increase during the first year? During the second year? What about in during the $n-$th year?

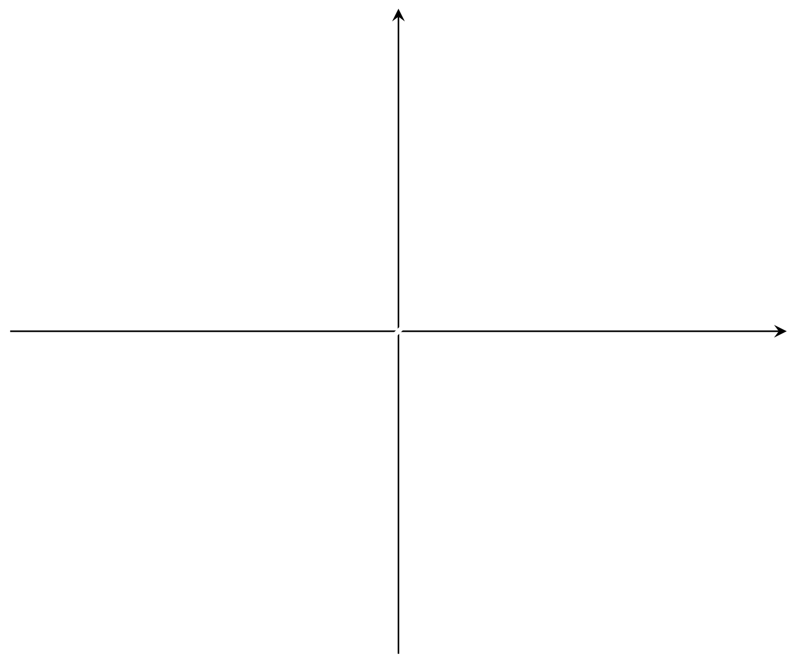
(b) Investment plan version 2:

i. Suppose Kyle's investment grows by 10% per year. Make a table that shows the value of his investment over the course of the next 5 years.

ii. Write down a function $g(x)$ that gives the balance in his bank account x years after his initial deposit.

2. In a certain town in 1975, the amount of whole milk consumed by the average resident was 20 gallons (over the entire year). Since 1975, that consumption has been decreasing at a rate of 2% per year. Write a formula for $M(t)$, the amount of milk consumed by the average resident in year t, where t is measured in years since 1975.

3. On the following set of axes, graph $y = 2^x$, $y = 3^x$, and $y = \left(\frac{1}{2}\right)^x$. Label all intercepts and asymptotes. (Also indicate which graph is which.)



4. If $f(x) = 2^x$, what can you say about the signs of $f'(x)$ and $f''(x)$?

---

5. Dolby is having a hard time remembering his exponential rules. He can't remember which of the rules below, if any, are correct. Identify the correct rules and help give Dolby by giving him a strategy to decide which rules are correct.

   (a) $x^a x^b = x^{ab}$

   (b) $x^a x^b = x^{a+b}$

   (c) $(x^a)^b = x^{ab}$

   (d) $(x^a)^b = x^{a+b}$

6. Zoe is growing some bacteria in a petri dish. Suppose the petri dish starts with 100 bacteria, and the number of bacteria doubles each hour - i.e. increases by 100%. Let $B(t)$ be the number of bacteria in the petri dish after t hours.

   (a) Fill out the following table:

t	$B(t)$ (bacteria after t hours)
0	100
1	
2	
3	
4	
5	

   (b) Write a formula for $B(t)$.

18 Applications of Exponentials

1. **Finance**

   (a) Marius decides to open a bank account with an opening deposit of \$1000. Suppose that the account earns a nominal annual interest rate of 6%, compounded annually.[6] Assuming Marius completely ignores the account after he opens it (so he doesn't make any deposits or withdrawals), how much money does the account have t years after Marius opens it? Can you graph his money vs. time?

   (b) Suppose that Marius had instead deposited his money in a bank that offered quarterly compounding (but everything else was the same). That is, the bank offers a 6% nominal annual interest rate with 4 compounding periods a year. How much money would the account have after t years?

   (c) In this account, what is the percent increase in money per year? (This is called the *annual percentage yield* of the account.)

   (d) What if Marius had used a bank that offered n compounding periods a year? How much money would the account have after t years, and what would be the annual percentage yield?

[6] For economists, there are two uses of the phrase "nominal interest rate"; one is trying to account for the effects of inflation. We're using it in the other sense, which will be easier to understand a bit later.

2. **Chemistry.** The half-life of the radioactive isotope radium-226 is approximately 1600 years.

   (a) Suppose you have a sample of radium-226 whose size in mg is currently S_0. Write a formula for the amount of radium-226 that remains in the sample at time t, where t is measured in centuries and $t = 0$ means the current time.

   (b) Suppose you have a sample of radium-226. By what percent does the amount of radium-226 in the sample decrease per century?

   (c) How many years would it take a 100 mg sample of radium-226 to decay to 25 mg?

   (d) About how many years would it take a 100 mg sample of radium-226 to decay to 5 mg?

3. **Sociology.** The population in a certain area of the country is increasing. In 1990 the population was 100,000, and by 2010 it was 200,000.

   (a) If the population has been increasing linearly, was the population in 2000 equal to 150,000, greater than 150,000, or less than 150,000? Explain your reasoning.

   (b) If the population has been increasing exponentially, was the population in 2000 equal to 150,000, greater than 150,000, or less than 150,000? Explain your reasoning.

   (c) If the population has been increasing exponentially and continues to do so, what do you expect the population to be in 2020?

   (d) If population is increasing linearly, in what year will there be a million people in this geographic region?

   (e) If population is increasing exponentially, in what year will there be a million people in this geographic region?

4. **Biology and Public Health** *Escherichia coli*, better known as E. coli, is a bacterium that can cause peritonitis, a potentially fatal disease. E. coli is sometimes found in ground beef, so food safety inspectors would like to be able to test for it. The challenge is to be able to take a tiny sample of ground beef (so that the rest can be sold to customers) and check the sample for E. coli, even if the sample has just a single bacterium.

   However, DNA tests are not sensitive enough to detect just a single bacterium, so the solution food inspectors have is to put the ground beef sample in "a broth infused with nutrients that E. coli likes to eat, put in a warm place to rest for 10 hours" [(7)]. Let's understand why. Under such conditions, it's estimated that a population of *E. coli* doubles every 30 minutes. Suppose that at time $t = 0$, t measured in minutes, the sample has just a single E. coli bacterium.

   (a) First, make a table for the number of bacteria:

t	# of bacteria after t minutes
0	

   (b) Write a formula for the number of *E. coli* present t minutes later.

   (c) Draw a graph of the number of *E. coli* t minutes later.

   (d) How many bacteria are there after 10 hours?

   (e) Write a formula for the number of *E. coli* present t hours later.

[(7)] (http://www.nytimes.com/2015/04/05/business/private-eyes-in-the-grocery-aisles.html)

19 The Derivative of an Exponential Function

1.

$f(x) = 2^x$

x	$f(x)$	estimate of $f'(x)$
0	1	0.693
1	2	1.386
2	4	2.774
3	8	5.547
4	16	11.094

$g(x) = 3^x$

x	$g(x)$	estimate of $g'(x)$
0	1	1.099
1	3	3.298
2	9	9.893
3	27	29.679
4	81	89.020

$h(x) = 10^x$

x	$h(x)$	estimate of $h'(x)$
0	1	2.305
1	10	23.052
2	100	230.524
3	1000	2305.238
4	10,000	23052.381

What do you observe?[(8)]

2. Let $f(x) = 2^x$.

   (a) Sketch a graph of $f'(x)$.

   (b) Using the limit definition of the derivative, write a limit that is equal to $f'(0)$.

   (c) Using the limit definition of the derivative, write a limit that is equal to $f'(x)$. How does this relate to $f'(0)$?

[(8)] If you had just done the case 2^x this might have been more challenging!

3. (a) If $f(x) = b^x$ (where b is a positive constant), what can you say about $f'(x)$?

(b) Suppose $g(x)$ is an exponential function; that is, $g(x) = C \cdot b^x$ for some C and some $b > 0$. What can you say about the relationship between $g'(x)$ and $g(x)$?

Let A and B be two changing quantities.

We say that A and B are **proportional** to one another if one is always a constant multiple of the other, i.e.

$$A = kB$$

for some constant k (called the *constant of proportionality*).

4. (a) How do we define e?

   (b) About how big is e?

   (c) If k is any constant, what is $\dfrac{d}{dx}\left(e^{kx}\right)$?

5. Use the tangent line approximation of e^x at $x = 0$ to approximate $e^{0.1}$. Is your estimate an over estimate or an underestimate?

20 Exponential Function Wrap Up

1. You have recently published a video to YouTube. When there are 80,000 views, you notice that rate at which people are viewing your video is 1000 people per day. Later when your video has 100,000 views, you notice that the rate at which people are viewing your video is 5000 people per day. If you wanted to model how many views your video had as a function of time, would a linear model or an exponential model be a good choice? Explain your answer.

2. Find an exponential function whose graph through the points $(4, 8)$ and $(6, 10)$.

3. A population of bacteria in a petri dish is growing exponentially. When there are 8000 bacteria, the population is growing at a rate of 400 bacteria per hour. How quickly (in bacteria/hr) is the bacteria population growing when there are 10,000 bacteria in the dish?

   If there were 1000 bacteria at time $t = 0$ (with t in hours), which of the following equations for $P(t)$ is correct?

   (a) $P(t) = 1000e^{20t}$ (b) $P(t) = 1000e^{\frac{t}{20}}$ (c) $P(t) = 50e^{t}$ (d) $P(t) = 1000\left(\frac{e}{20}\right)^{t}$

4. The function $f(x) = \dfrac{3x}{e^x}$ is an example of a surge function. Surge functions are used to model many situations. For instance, the amount of an IV drug in the blood x minutes after the drug is administered can be modeled as a surge function.

   (a) Compute $f'(x)$.

   (b) For what values of x is $f'(x)$ positive? For what values of x is $f'(x)$ negative?

   (c) For what values of x is $f(x)$ concave up? For what values is it concave down? Give exact answers.

   (d) Find $\lim\limits_{x\to-\infty} f(x)$.

   (e) Use your answers to the previous parts to sketch a graph of $f(x)$.

5. A population of mosquitoes is growing exponentially. When there are 5000 mosquitoes, the mosquito population is growing at a rate of 100 mosquitoes per month.

   (a) We know that the mosquito population is changing at a rate proportional to its size. What is the constant of proportionality in this case? (i.e. fill in the blank: $M'(t) =$ ___________ $M(t)$.)

   (b) How fast is the population growing when there are 6000 mosquitoes?

   (It is not necessary to find an equation for $M(t)$ in order to solve this problem - in fact, you haven't been given enough information to find $M(t)$!)

   (c) Let t be measured in months, and suppose that $M(0) = 3000$. Which of the following is a valid equation for $M(t)$?

      i. $3000e^{0.02t}$ ii. $3000e^{100t}$ iii. $3000e^{\frac{3t}{4}}$ iv. $3000e^{\frac{t}{100}}$

   Explain your reasoning by choosing the argument that is correct:

   Ginny: Since we know that there are 5000 mosquitoes when the growth rate is 100 mosquitoes per month, the proportionality constant is 100 and this gives the general form $M(t) = 3000e^{100t}$.

   Luna: The proportionality constant in #1 is 0.02 and this gives the general form $M(t) = 3000e^{0.02t}$.

   Fred: We divide by the growth rate 100 mosquitoes per month and so $M(t) = 3000e^{\frac{t}{100}}$.

21 Constant Acceleration

1. On Earth, acceleration due to gravity is approximately 9.8 m/s^2, or 32 ft/s^2. One day, Katie leans out her dorm window (a height of 64 ft above the ground) and hurls an apple straight up in the air with an initial velocity of 48 ft/s.

   (a) Write a function $h(t)$ which gives the height of the apple in feet t seconds after Katie tosses it.

   (b) What is the greatest height that the apple reaches?

2. Angus is selling tickets for the Kroks upcoming a cappella event. From previous months, he knows that when the price of tickets was \$10, 99 tickets were sold and when the price of tickets was \$7 then 126 tickets were sold. Suppose Angus wants to model how many tickets the Kroks will sell based on price of tickets, using a function $T(x)$, where T is number of tickets sold when the price is x dollars. Assuming T is linear, it is given by the function $T(x) = -9x + 189$.

   (a) What is $T'(x)$? Is it positive or negative? Explain why this makes sense in terms of the model.

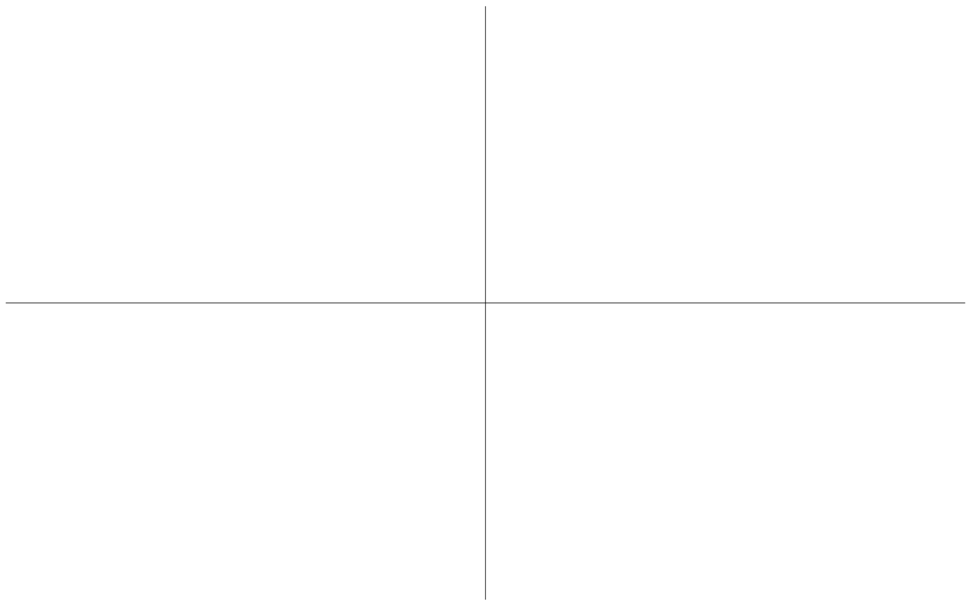
   (b) Find a formula for $A(x)$, the revenue from ticket sales, if the price of a ticket is x dollars.

   (c) Graph your revenue function, labeling all intercepts.

   (d) From your graph, determine the price that maximizes revenue.

   (e) Find $A'(x)$. What does it mean in terms of the model? Graph it.

3. (a) Sketch the graph of $y = -2(x-1)^2 + 18$.



(b) Does the graph have x-intercepts? If so, where?

4. Sketch the following parabolas, and say where the vertex is.

(a) $y = 2(x-1)(x-5)$

(b) $y = -2x^2 + 4x - 7$

5. In each part, is there enough information given to determine the equation of the parabola or quadratic function? If so, write the equation.

   (a) A parabola passing through the points $(0, 3)$, $(-1, 6)$, and $(2, 9)$.

   (b) A quadratic function whose derivative is $2x + 3$.

   (c) A parabola with x-intercepts at $x = 1$ and $x = 4$ and y-intercept at $y = -1$.

   (d) A parabola with vertex at $(2, -4)$ and an x-intercept at $x = 5$.

22 Cubics and Higher Degree Polynomials

1. For each description below, sketch a graph of a cubic function $f(x)$ with the characteristic(s) specified and find a formula for $f(x)$.

   (a) $f(x)$ has zeros at $x = -\pi$, $x = 2$, and $x = 5$. How many possible answers are there?

   (b) $f(x)$ has only two zeroes: at $x = 2$ and $x = 5$ and a y-intercept of -4. How many possible answers are there?

   (c) $f(x)$ has only one zero, at $x = 2$, and passes through the point (1,5). How many possible answers are there?

   (d) Graph $y = x(x + 2)(x - 2) = x^3 - 4x$. Are the tangent lines at $x = 1$ and $x = -1$ horizontal?

2. Let $C(x) = 2x^3 - 3x^2 - 12x$.

   $C(x)$ has three roots: one at $x = 0$, one at $x \approx -1.8$, and one at $x \approx 3.3$.

   Find $C'(x)$, and use it to help graph $C(x)$. How do the graphs relate?

3. Find a cubic polynomial with horizontal tangents at $x = 5$ and $x = 2$. How many possible answers are there? Is $x = 5$ and $x = 2$ always a turning point?

4. For each description below, sketch a graph of a polynomial function $P(x)$ with the characteristic(s) specified <u>if it is possible!</u> Also, find a formula for $P(x)$.

   (a) A degree 5 polynomial with no zeros and $\lim_{x\to\infty} P(x) = -\infty$. How many possible answers are there?

   (b) A degree 6 polynomial with no zeros and $\lim_{x\to\infty} P(x) = -\infty$. How many possible answers are there?

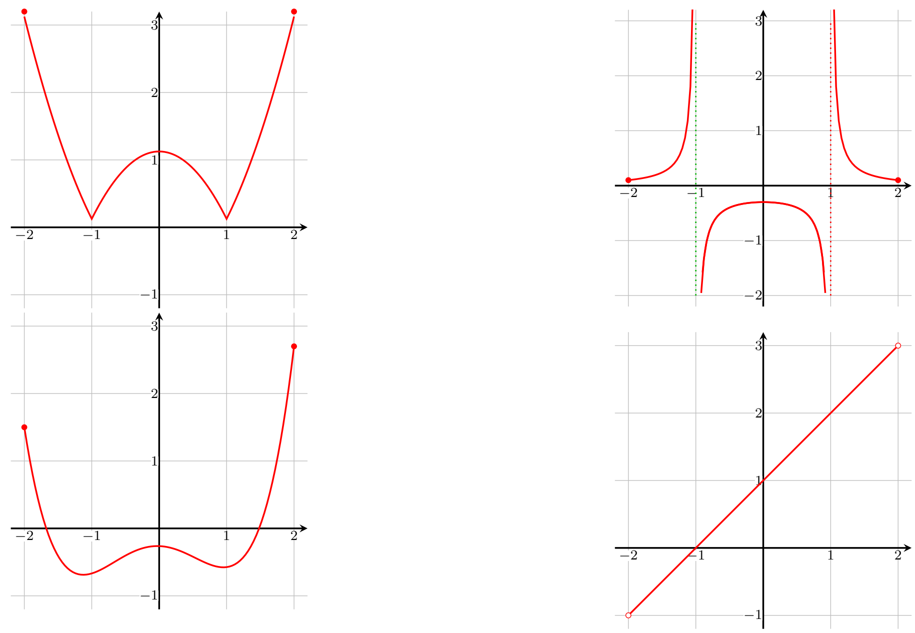
5. How many zeros can a degree n polynomial have? How few?

   How many turning points can a degree n polynomial have? How few?

23 Analysis of Local Extrema

1. Intuitively, a local maximum is a point which is at least as high as all points around it. More precisely, f has a local maximum at c if $f(c) \geq f(x)$ for all x in the domain near c. How would you define a local minimum?

   Mark the local maxima and local minima on the graphs below.



2. Often, our goal is to find the absolute minimum or maximum (if it exists) of a function on a given interval. A good starting point is to first find the local extrema.

   (a) How would you create a list of possible local extrema?

   (b) How would you go about deciding if a critical point is a local max or a local min?

3. In this problem, we'll look at the cubic function $f(x) = x^3 + 3x^2 + 1$.

   (a) Find all critical numbers of f.

   (b) Make a sign chart for f', and use this to decide whether each of the critical points you found is a local minimum, a local maximum, or neither.

   What you just did is called the ***First Derivative Test*** *to decide whether a critical point is a local minimum, a local maximum, or neither.*

4. Let $f(x) = x^4 - 8x^2 + 16$.

   (a) Find all critical numbers of f.

   (b) For each critical number c of f, find the sign of $f''(c)$. What does this tell you about the critical number c?

5. Let $f(x) = x^3$ and $g(x) = x^4$.

   (a) Sketch graphs of $f(x)$ and $g(x)$.

   (b) What is $f'(0)$? $f''(0)$?

   (c) What is $g'(0)$? $g''(0)$?

Second Derivative Test. Suppose $f'(c) = 0$. What can you conclude if you have the following additional information about $f''(c)$?

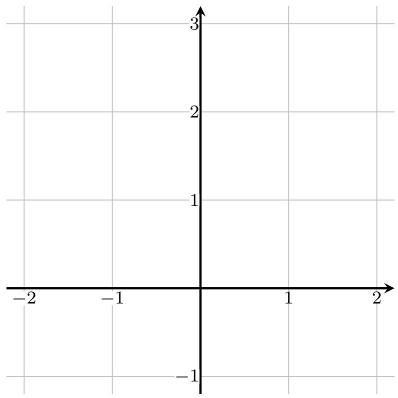
- $f''(c) > 0$.
- $f''(c) < 0$.
- $f''(c) = 0$.

6. Let $f(x) = 3x^{1/3} + 4x$. Find all critical numbers of f, and determine whether each critical number is a local minimum, local maximum, or neither.

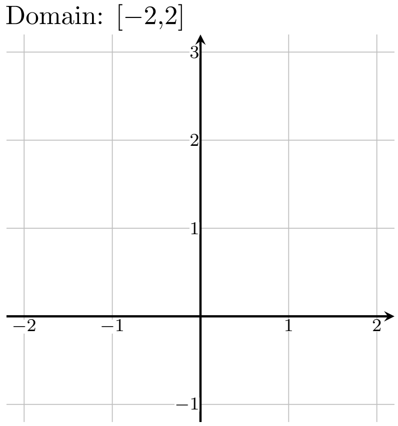
24 Analysis of Absolute Extrema

1. If possible, create graphs of functions satisfying each description:

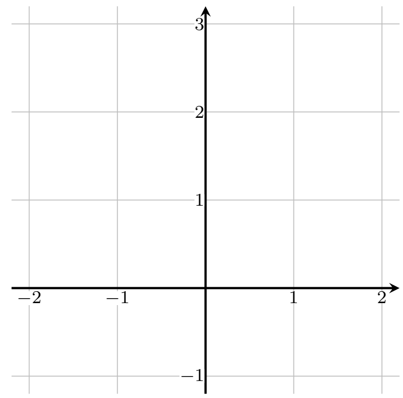
   (a) A continuous function with an absolute maximum of 3 and no absolute minimum.
   Domain: [−2,2)



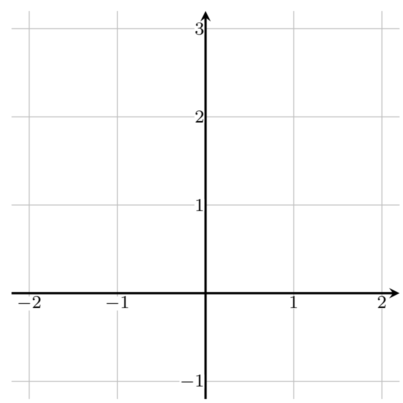
   (b) A function with no absolute maximum and no absolute minimum .
   Domain: [−2,2]



   (c) A continuous function with no absolute maximum and no absolute minimum.
   Domain: (−2,2)

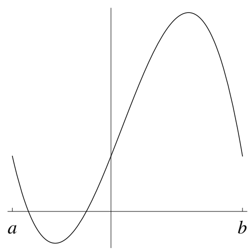


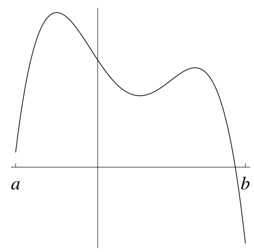
   (d) A continuous function with no absolute maximum and no absolute minimum.
   Domain: [−2,2]

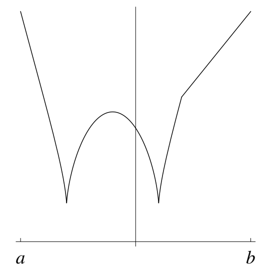


2. When can you be guaranteed that a function $f(x)$ has an absolute max and an absolute minimum?

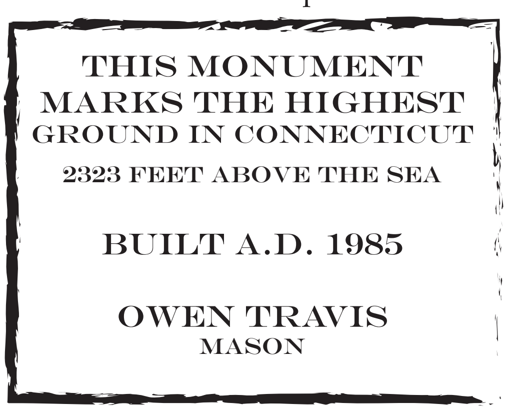
3. We now know a continuous function defined on a closed interval will have an absolute maximum and an absolute minimum. Often, our goal is to find the absolute minimum or maximum (if it exists) of a function on a given interval. Look at the graphs below and think about a strategy. How would you go about identifying absolute maximum and absolute minimum on $[a,b]$?



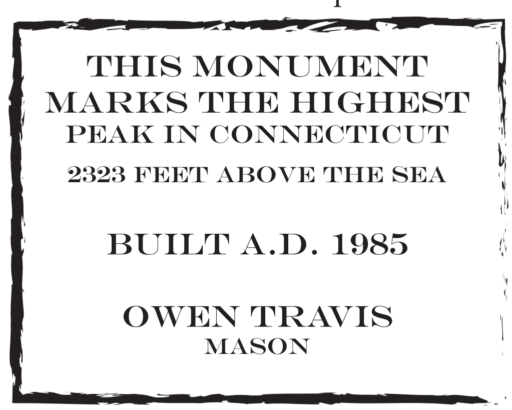




Acutal Inscription



A Correct Description



4. Suppose $f(x)$ is a continuous function defined on a closed interval. Is every critical point an absolute maximum or an absolute minimum?

5. (a) Does $f(r) = 2\pi r^2 + \frac{256\pi}{r}$ have an absolute maximum and absolute minimum on $[1, 8]$? If so, where? (Give the r-values at which the absolute minimum and absolute maximum occur.)

   (b) Does f have an absolute minimum and absolute maximum on $(0, \infty)$? If so, where?

6. A soda company wants its aluminum soft drink cans to have a volume of 128π cubic centimeters. The company's factory can only manufacture cans that are at least 2 cm tall and have a radius of at least 1 cm. In order to conserve resources, the company wants to minimize the amount of aluminum needed for a single can. What dimensions should they make their cans?

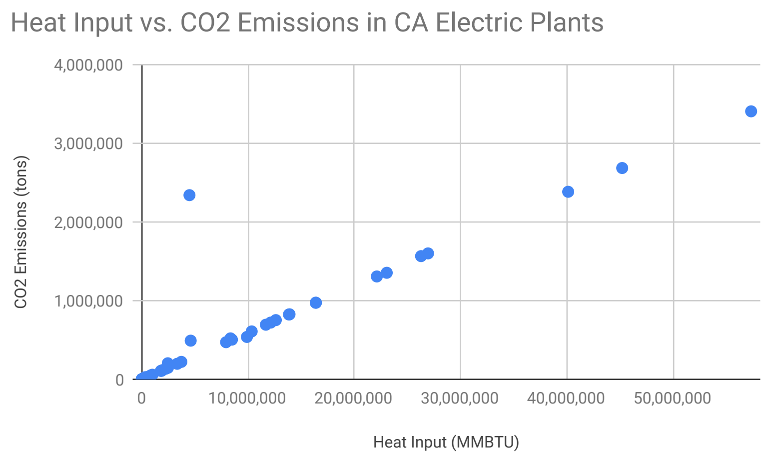
25 Optimization

1. For the past couple of days, we've been talking about how to find the global (also known as absolute) maximum and minimum of a given function $f(x)$ on a given domain, or to show that the function has no global maximum or minimum on the domain. Summarize the methods we learned.

2. A farmer has 40 feet of fencing, and he wants to fence off a rectangular pen next to his barn. The barn will be one side of the pen, so that side needs no fencing. In order for the cow to be able to turn around in the pen, the pen needs to be at least 5 feet long and 5 feet wide. What is the largest area the pen could have?

3. The soda company in our problem yesterday realizes that they need to use stronger aluminum for the tops and bottoms of the cans, and this stronger aluminum costs 3 times as much per square inch as the aluminum used for the sides. If the company wants to minimize the cost of each can, what dimensions should their cans be? Remember the volume of the can is 128π cubic centimeters.

4. The following graph(9) shows the heat input in MMBTU (the energy industry's way of writing millions of BTUs) and the CO2 output (in tons) for most electricity generation plants in California.



(a) Does the data represent a linear relationship? How do you know?

(b) We can use optimization techniques to find a line that would best fit this data. As a first example of fitting a model to a given data set, find a linear model $f(x) = mx$ that minimizes the error in the following data set.

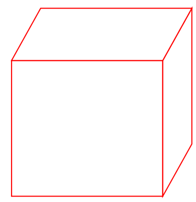
$$\{(1,1),(2,2),(3,2)\}$$

(9) Emissions and Generation Resource Integrated Database

26 More Optimization

1. You need to make a poster for a poster session. The organizers of the session insist that your poster has to be 200 square inches and will have 1 inch margins on the sides, a 3 inch margin on the bottom and a 1 inch margin on the top. What dimensions will give the greatest printed area?

2. Your architectural firm is designing a building shaped like a box with a **square base**, tiled on the **sides** and **top** with solar panels. (There are no solar panels on the bottom of the building.) The building will have a volume of 8 cubic hectometers (hm). Zoning regulations require that the base of the building have area no less than 1 hm^2 and no more than 9 hm^2. Furthermore, the solar panels for the top have been engineered to absorb twice as much solar energy per hm^2 as the ones for the side.



Find the dimensions of the building that maximize solar energy absorption.

3. The number of tweets per hour about getting the vote out can be modeled by the function

$$f(t) = 100te^{-.12t}$$

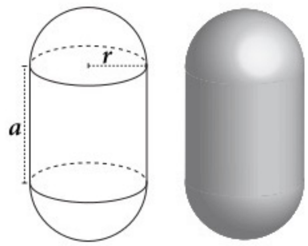
where t is the number of months after March 1st and $f(t)$ gives is the number of thousand tweets per hour. *#getoutthevote*

(a) Draw a graph for $f(t)$

(b) When are the most tweets about getting the vote out happening?

4. The Phillit Pharmaceutical Company is designing a capsule for a new drug. The capsule is formed by capping a cylinder with two half-spheres, one at each end. Justify your answer.

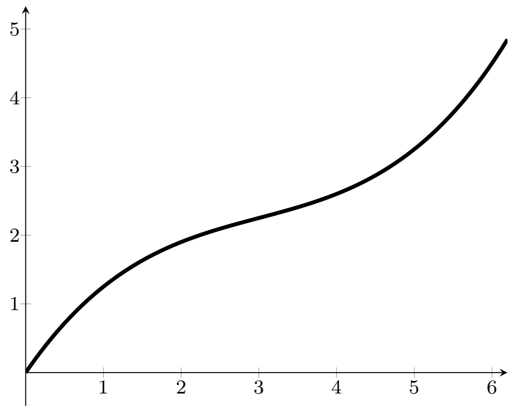
   The manufacturing division has passed a few constraints along to the design team for the pill's design. The cylindrical band around the middle (shown below with height a) should have an area of exactly 60π mm^2, the radius should be at least 3 mm, and the height of the cylinder should be at least 5 mm. With these constraints, what are the radius and height that would maximize volume? [Note: the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.]



5. A company is manufacturing smartphones at a cost of \$100 per phone. They are trying to decide how to price their phones to maximize profit. Their market research suggests that, if they price the phones at \$200 per phone, then they will sell 10,000 phones a month. For each \$5 increase in price, the number of phones they will sell decreases by 3%.

27 Inverse Functions

1. Last weekend, Brianna went for a long leisurely walk around Boston. Here is the graph of $f(t)$, her distance in miles t hours after the start of her walk.



(a) How long did it take Brianna to walk 2 miles?

(b) How long did it take Brianna to walk 4 miles?

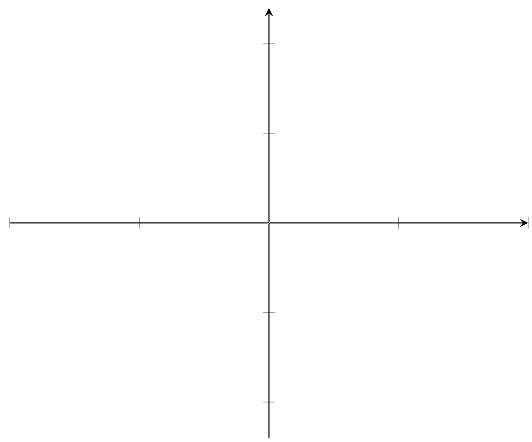
(c) What does the function f^{-1}, the inverse function of f, represent? Can you sketch it?

(d) How do the domain and range of f^{-1} relate to the domain and range of f?

2. Let $f(x) = x^3$.

   (a) What is $f^{-1}(x)$?

   (b) Sketch $f(x)$ and $f^{-1}(x)$ on the same set of axes.

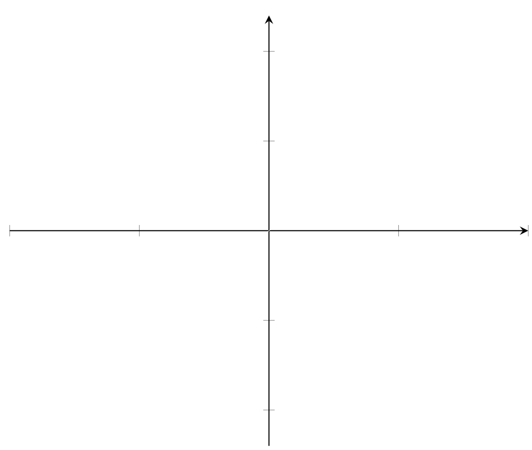


   (c) What is $f(x)^{-1}$? Is it the same as $f^{-1}(x)$?

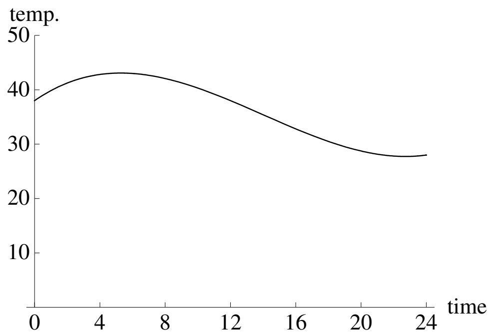
3. Let $g(x) = x^2$.

   (a) What is $g^{-1}(x)$?

   (b) Sketch $g(x)$ and $g^{-1}(x)$ on the same set of axes.



4. Suppose the following graph shows the temperature at Logan Airport over a particular 24-hour period. Let $f(t)$ be the temperature at time t. Is f invertible? How can you tell?



5. Decide whether each of the following functions is invertible. If so, what is the domain of the inverse function?

   (a) $f(x) = 2^x - 1$.

   (b) $g(x) = -2x^6 + 7x^5 - \pi x + \sqrt{3}$.

   (c) $h(x) = x^3 + x$.

   (d) $r(x) = \sqrt{x+5}$

28 Undoing Exponential Functions

1. When a new technology (like iPods, Facebook, Twitter, etc.) starts to become popular, it often experiences exponential growth. In 2007, Facebook reported that the number of Facebook members was growing exponentially by about 3% per week. On October 25 of that year, Facebook reached 50 million members; if the growth trend continued, when did the number of members reach 100 million?

2. Numbers can also be written as powers of 10. We call this, "Super Scientific Notation"

   (a) Explain why 2500 must be power between 10 with an exponent between 3 and 4.

   (b) Using trial and error on your calculator, find the exponent $\square$ such that $10^{\square} = 2500$. How accurate can you be?

3. Express the numbers in the table super-scientific notation. Try to be accurate to at least 2 decimal places.

Number	Number in S-SN	Exponent from S-SN
1		
10		
333		
-2500		
314156		
1000		

4. We can extend this idea and rewrite numbers as powers of other numbers. Using your calculator, estimate the exponent to which we raise 3 to get 1200.

> The **logarithm** $\log_b(x)$ is the exponent to which we raise b to get x.
>
> - We require $b > 0$ and $b \neq 1$.
> - We say $\log_b(x)$ aloud as "log base b of x."

5. Evaluate each of the following logarithms by using the relationship between exponents and logarithms.

   (a) $\log_3(27)$ (b) $\log(10000)$ (c) $\log_2\left(\frac{1}{8}\right)$

(a) $\log_2 32$

(d) $\log \dfrac{1}{100}$

(g) $\log_{1/2} \dfrac{1}{8}$

(b) $\log_5 25$

(e) $7^{\log_7 3}$

(h) $5^{\log_5 11}$

(c) $\log_3 \dfrac{1}{9}$

(f) $\log_7(1)$

(i) $9^{\log_3(4)}$

6. (a) Make a table for the function $\log_7 x$.

(b) Graph $\log_7 x$.

What is the domain of $\log_7 x$? What are its intercepts and asymptotes?

We call the base e logarithm the "natural logarithm," and write "$\log_e x$" as $\ln x$.

"$\ln x$" is the power to which e must be raised in order to get x.

So, if $e^k = b$, $\ln b = k$.

7. Sketch the graphs of $y = e^x$ and $y = \ln x$ on the same axes.

8. Use the fact that $\frac{d}{dx}e^{kx} = ke^{kx}$ to calculate $\frac{d}{dx}b^x$.

29 Logarithm Rules

1. Simplify each as much as possible.

   (a) $\log_2 8$

   (b) $\log_5 25$

   (c) $\log_3 \frac{1}{9}$

   (d) $\log \frac{1}{10\sqrt{10}}$

   (e) $\ln\left(e^{\pi}\right)$

   (f) $7^{\log_7 3}$

   (g) $11^{\log_5 11}$

   (h) $e^{-3\ln 2}$

   (i) $\log_3\left(3^4 \cdot 3^7\right)$

   (j) $\log_2\left(\frac{2^6}{2^{11}}\right)$

2. Express $\log(u^2 w)$ in terms of $A = \log u$ and $B = \log w$.

3. Express $\log_b\left(\frac{u^3}{w^2}\right)$ in terms of $A = \log_b u$ and $B = \log_b w$.

4. If possible, simplify each expression, and write it in terms of $A = \log_2 u$ and $B = \log_2 w$.

   (a) $\log_2\left(\frac{2}{\sqrt{uw}}\right)$.

   (b) $\log_2\left(u + \sqrt{2w}\right)$.

5. Let $x = \log_5 R$. Express $\log_5(R^p)$ in terms of x.

6. Let $y = \log_2 M$ and $z = \log_2 N$. Express $\log_2 MN$ in terms of y and z.

7. Let $y = \log_b M$ and $z = \log_b N$. Express $\log_b \frac{M}{N}$ in terms of y and z.

8. $\log_2 \left(u + \sqrt{2w}\right)$.

Let $b^x = Q$ and $b^y = R$.

Using exponential rules that we already know, we get the following rules about logarithms:

Rule for Exponentials	**Rule for Logarithms**
$b^x b^y =$	____________________
$\dfrac{b^x}{b^y} =$	____________________
$(b^x)^p =$	____________________

Be Careful! We *don't* have rules for simplifying $\log_b(P + Q)$ or $(\log_b P)(\log_b Q)$.

9. Let $\log 2 = a$ and $\log 3 = b$. Express each of the following in term of a and b. There should be no logarithms explicitly in the expression you give.

   (a) $5 \log \dfrac{2}{3}$ (b) $5 \log \sqrt[3]{6}$ (c) $\log \dfrac{16}{\sqrt[4]{9}}$

10. Simplify the following:

    (a) $\log_2(120) - \log_2(3)$ (b) $\log_8 x + \log_8(x + 1)$ (c) $\log_5(7^3) - \log_5(7)$

11. On the same set of axes, graph $\ln x$, $\log_2 x$, and $\log_3 x$.

12. Solve the equation $5^x = 17$ two ways, once by taking $\log_5$ of both sides and once by taking ln of both sides. (The first step of each method is given below.)

$5^x = 17$	$5^x = 17$
$\log_5 (5^x) = \log_5 17$	$\ln (5^x) = \ln 17$

13. (a) Looking at your answer to #12, guess a formula for $\log_b a$ in terms of natural logs.

(b) Show that your formula must be true in general. (This formula is known as the change of base formula.)

(c) Explain how the change of base formula is consistent with your graphs in #11.

14. Rewrite the following in terms of the natural log:

(a) $\log_6 14$ (b) $\log_{12} 3$ (c) $\log_4 8$

15. Graph $f(x) = \log_5\left(\frac{25}{x^2}\right) + \log_5 x$. (*Hint:* First use log rules to simplify.) Label all intercepts and asymptotes, and label at least two other points on the graph with exact coordinates.

30 Derivatives of Logarithms

1. Let $f(x) = \ln x$.

   (a) Sketch the graph of $f(x)$.

   (b) Use your sketch of $f(x)$ to sketch the graph of $f'(x)$.

2. How can the derivative of $\ln x$ be approximated?

x	$\frac{d}{dx} \ln x$ (approx)
$\frac{1}{10}$	9.53
$\frac{1}{2}$	1.98
1	0.995
2	0.499
3	0.333
4	0.250
5	0.200

Conjecture: The derivative of $\ln x$ is

We will take your conjecture as fact and prove this later.

3. What is the derivative of $\log_7 x$?

4. What is the derivative of $\log_b x$?

Summary:

- $\dfrac{d}{dx}(x^n) =$ (n is a constant)
- $\dfrac{d}{dx}(e^{kx}) =$ (k is a constant)
- $\dfrac{d}{dx}(b^x) =$ ($b > 0$ and constant)
- $\dfrac{d}{dx}(\log_b x) =$ ($b > 0$, constant)
- $\dfrac{d}{dx}(e^x) =$
- $\dfrac{d}{dx}(\ln x) =$

5. Find the derivative of each of the following functions.

(a) $y = \ln(3x)$

(b) $f(x) = x \ln x$.

(c) $y = \frac{1}{5}\log_3(x) + \sqrt{x}$.

(d) $y = \dfrac{\sqrt[3]{5^x}}{7}$.

To compute derivatives of more complicated functions we need more logarithmic properties:

- $\log_b b^\star =$ ______
- Write $\log_b R = y$ in exponential form: ____________________
- $b^{\log_b \star} =$ ______
- $\log_b(QR) =$ ____
- $\log_b(Q/R) =$ ____
- $\log_b(R^p) =$ ____

6. Does $f(x) = x \ln \sqrt{x}$ have a global maximum value? A global minimum value? If the function has either of these, at what value of x is it attained? (Look back at #5(b).)

31 Solving Equations

In the problems below, solve each equation for x. When you use a log rule to simplify, say which rule you've used.

7. (a) $7^x = 12$ (b) $x^7 = 12$ (c) $\log_7 x = 12$

   (d) $7x + 12 = 3x + 4$ (e) $7x^2 + 12x = 0$

8. (a) $7^{4x+3} = 12$ (b) $5 \cdot 7^{x+3} = 7^{2x+1}$ (c) $7^x - \dfrac{49}{7^{x^2}} = 0$

9. (a) $4^{x+3} = 5^{2x-1}$ (b) $7 \cdot 4^{x+2} = \dfrac{5}{3^{2x+1}}$ (c) $2^{x^2} = 5 \cdot 3^x$

10. $2^{2x+3} = 8^{x-7}$.

11. $3^x \cdot \dfrac{5}{3^{x+1}} = 0$.

12. (a) $e^x(e^x - 5) = 6$.

(b) $2e^{2x} + 6 = 7e^x$.

13. Solve for x

(a) $2\log_3(x) = \log_3(4) + \log_3(x-1)$

(b) $\log_5(x^2) - \log_5(x) + \log_5(x^4) = 10$

32 Applications of Logarithms

1. When you take a couple of Tylenol tablets, the active ingredient (acetaminophen) is absorbed into your system and then gradually dissipates. Though individual responses may vary, typically acetaminophen has a half-life of around 180 minutes[10].

   (a) Sketch a graph of the amount of acetaminophen in your system as a function of time.

   (b) Construct a model $M(t)$ that describes the amount of amount of acetaminophen in your system t minutes after taking two regular strength tablets (600 mg total acetaminophen).

   (c) The parameters in this problem vary from patient to patient. Describe how the following changes would affect your model:

      i. You increase the initial dose to 1200 mg.

      ii. The half-life of a patient is 216 minutes.

   (d) From clinical trials it is known that the amount of acetaminophen needed in your system to receive any therapeutic effect is 200 mg. About how long after you take a two regular strength tablets does the therapeutic effect wear off?

[10] This problem and the next problem are adapted from a blog called Mathalicious

(e) Find the inverse function of $M(t)$. What does this function describe?

(f) Right after you take two regular strength Tylenol how much time needs to pass in order for 1 mg of acetaminophen to dissipate?

(g) Calculate $M'(t)$. What does this represent?

(h) Calculate the derivative of the inverse of $M(t)$. What does this represent?

(i) Clinical trials have shown that having high levels of acetaminophen in your system for long periods of time can cause liver damage. What would a patients half-life parameter have to be such that after taking 1000 mg of acetaminophen 400 mg remains in the system for 6 hours?

2. If a little bit of Tylenol is therapeutic, a lot must be even more so, right? Well, not quite. While your pounding head may love a dose of acetaminophen, your liver is less of a fan, and too much acetaminophen can cause it to shut down. Also, unlike other pain relief drugs, the difference between a healthy dose of acetaminophen and a dangerous dose can be as small as two extra-strength Tylenol tablets (1000 mg acetaminophen).

   Suppose you've got a killer headache, and take two extra-strength Tylenol tablets every 6 hours for an entire day. Whether or not this is dangerous depends on your individual tolerance, as well as whether or not you're taking any other drugs containing acetaminophen.

   (a) Draw a graph of $D(t)$, a function that describes the amount of acetaminophen in your system t minutes after taking two extra-strength Tylenol tablets (1000 mg total) every 6 hours for an entire day.

   (b) Is $D(t)$ invertible? How do you know?

   (c) What is the maximum amount of acetaminophen in your system during those 24 hours?

   (d) If you continued taking two tablets every 6 hours forever, what would be the maximum amount of acetaminophen in your system?

   (e) When there is 1200 mg of acetaminophen in your system, about how long does it take to dissipate 1 mg of acetaminophen?

3. Sketch a graph of $x \ln(x)$. Label all intercepts and local extrema.

33 The Chain Rule

1. Flavia has an empty flask. The flask is shaped so that, if she were to fill the flask with v mL of water, then the height of water in the flask would be v^2 mm.

   (a) Flavia starts pouring water into the flask. At 1 pm, she has poured exactly 10 mL of water into the flask. Can you tell what the instantaneous rate of change of height of water with respect to time is at 1 pm, or do you need more information?

   (b) Suppose that you are also told that, at 1 pm, Flavia is pouring water into the flask at an instantaneous rate of 2 mL/s. Now, can you tell what the instantaneous rate of change of height of water with respect to time is, or do you need more information?

2.

3. Fill in the following tables.

Table 1: Derivatives we knew before

$f(x) = x^n$	$f'(x) =$
$f(x) = e^{kx}$	$f'(x) =$
$f(x) = b^x$	$f'(x) =$
$f(x) = \ln x$	$f'(x) =$

Table 2: Derivatives we know now

$h(x) = [g(x)]^n$	$h'(x) =$
$h(x) = e^{g(x)}$	$h'(x) =$
$h(x) = b^{g(x)}$	$h'(x) =$
$h(x) = \ln[g(x)]$	$h'(x) =$

4. Differentiate each of the following functions.

   (a) $h(x) = \ln(7x^2 + 3)$

   (b) $q(t) = e^{t^3} - 12e^2 t$

   (c) $j(x) = [\ln(7x^2 + 3)]^5$

   (d) $h(x) = \pi^3(7x + 3)^{10}$

   (e) $y = \dfrac{\sqrt{x + \sqrt{x}}}{e}$.

   (f) $h(x) = e^{x^2} \cdot \ln(7x^2 + 3)$

5. How does the graph of $\ln(x + 5)$ relate to the graph of $\ln x$? Use the Chain Rule to differentiate $\ln(x + 5)$. Is your answer consistent with the graph and what you know about the derivative of $\ln x$?

34 The Chain Rule Continued

1. At time $t = 0$, Baloo starts blowing up a spherical balloon. His friend Bagheera observes that, at time $t = 3$, the balloon has a radius of 5 cm, and the instantaneous rate of change of the balloon's **radius** with respect to time is 2 cm per second (t is measured in seconds). At this time, what is the instantaneous rate of change of the balloon's **volume** with respect to time? (The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.) What are the units of your answer?

2. The price of tomatoes varies seasonally; suppose $P(t)$ gives the price of tomatoes in dollars per pound at time t, where t is measured in months since January 1, 2014. The demand for tomatoes (in other words, the amount of tomatoes people want to buy) depends on the price; say that $D(x)$ gives the demand for tomatoes when they cost \$$x$ per pound.

   What information would you need about P and D in order to find the instantaneous rate of change in demand for tomatoes with respect to time on March 1, 2014?

3. You have learned the Power Rule, which said that $\frac{d}{dx}x^n = nx^{n-1}$ for any real number n. However, we only *proved* this when n was either a positive integer or $\frac{1}{2}$. Now that we know the Chain Rule, we can finally prove the Power Rule for all real numbers n. In this problem, you'll use the Chain Rule to prove that the derivative of x^π is $\pi x^{\pi-1}$. (The same idea works for any exponent, not just π.)

   (a) Rewrite $f(x) = x^\pi$ as $f(x) = e^{\text{something}}$. What is the "something" in the exponent?

   (b) Now, use the Chain Rule and what you know about the derivative of e^x to show that $f'(x) = \pi x^{\pi-1}$.

4. Differentiate the following functions. (Don't try to simplify your answers.)

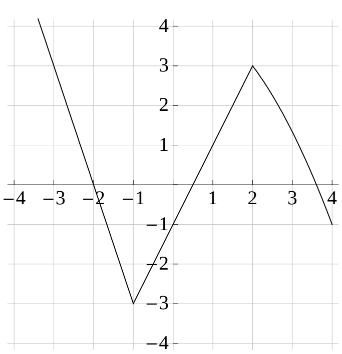
   (a) $h(x) = e^{3x} + x^6 \cdot \ln(7x^2 + 3)$

   (b) $j(u) = \ln(\ln u)$

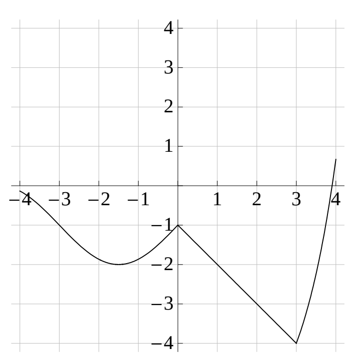
   (c) $f(x) = (x^2 + 3)^{\pi} \cdot \ln\left(4 - e^{3x^2 - 2}\right)$

5. Here are graphs of two functions, $f(x)$ and $g(x)$. If $F(x) = f(g(x))$, what is $F'(1)$?

$f(x)$



$g(x)$

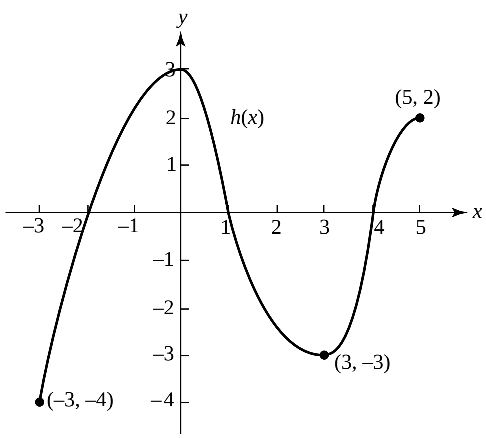


6. You are given the following information about three functions f, g, and h.

$$h(1) = 2 \qquad g(2) = 3 \qquad f'(3) = 6$$
$$h'(1) = 4 \qquad g'(2) = 5$$

If $r(x) = f(g(h(x)))$, do you have enough information to find $r'(1)$? If so, compute it. If not, what additional information do you need?

7. Here is the graph of a function $h(x)$.



(a) For which values of x is $h'(x)$ negative?

Define a new function $f(x)$ by $f(x) = [h(x)]^2$.

(b) Find all critical numbers of $f(x)$, and classify each as a local minimum, local maximum, or neither. (*Note:* This is asking about $f(x)$, not $h(x)$!)

(c) Does f achieve an absolute maximum on $[-3, 5]$? If so, where?

(d) Does f achieve an absolute minimum on $[-3, 5]$? If so, where?

35 Logarithmic Differentiation

1. Consider the function $f(x) = x^x$. It's neither an exponential function (it's not of the form b^x with b a constant) nor a power function (it's not of the form x^n with n a constant). You showed for homework that the derivative of $f(x)$ is neither $\ln x \cdot x^x$ nor $x \cdot x^{x-1}$. In this problem, we'll look at 2 ways of finding $f'(x)$.

   (a) **Method 1.** Express $f(x) = x^x$ as $e^{\text{something}}$, and then use the Chain Rule to find $f'(x)$.

   (b) Method 1 only makes sense when x^x is positive. What part goes wrong when $x^x \leq 0$? Should we worry about this?

   (c) **Method 2 (Logarithmic differentation).**

   Fill in the steps below. We start with the given equation

   $$f(x) = x^x.$$

   Take the natural log of both sides, and simplify the right side:

   Differentiate both sides with respect to x:

   Solve for $f'(x)$:

   (d) Do the two methods give you the same answer? Which do you like better?

Exponentiation Laws. Fill in as many blanks as you can!	**Logarithm Laws.** Fill in as many blanks as you can!
E1. $b^x + b^y =$ ______________________	L1. $\log_b(Q+R) =$ ______________________
E2. $b^{x+y} =$ ______________________	L2. $\log_b(Q-R) =$ ______________________
E3. $b^x - b^y =$ ______________________	L3. $\log_b(QR) =$ ______________________
E4. $b^{x-y} =$ ______________________	L4. $(\log_b Q)(\log_b R) =$ ______________________
E5. $b^{xy} =$ ______________________	L5. $\log_b\left(\frac{Q}{R}\right) =$ ______________________
	L6. $\log_b(Q^y) =$ ______________________

2. Find the derivative of each of the following functions.[11] Don't bother to simplify your answers.

   (a) $g(x) = (4x^2+5)^{e^x-1}$.

   (b) Find $\dfrac{dy}{dx}$ if $y = \dfrac{xe^x}{(x^2+2)^4(5x+2)^2}$, defined for $x > 0$.

[11] Technically, you should only use logarithmic differentiation on positive functions, since you can only take the log of a positive quantity. We won't worry about this technicality.

(c) $f(t) = 3^t + t^3 + t^{3t}$.

(d) $f(x) = (\ln x)^x - \sqrt{x^2+1}$

3. Suppose $Q(x) = \dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are positive for all x.

   (a) Use logarithmic differentiation to find $Q'(x)$.

   (b) As you can see, if you forget the Quotient Rule, you can always use logarithmic differentiation to figure it out again. What's another way to figure out the Quotient Rule if you've forgotten it?

4. Suppose you have four positive functions $f(x), g(x), h(x), j(x)$, and you want to find the derivative of the product $f(x)g(x)h(x)j(x)$. You *could* do this using the Product Rule, but it's much easier with logarithmic differentiation. Find the derivative of $f(x)g(x)h(x)j(x)$.

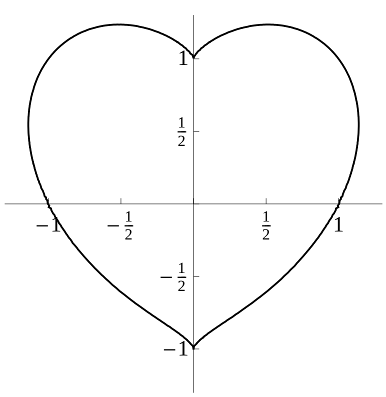
36 Implicit Differentiation

1. (a) If $g(x) = [f(x)]^3$, express $g'(x)$ in terms of $f(x)$ and $f'(x)$.

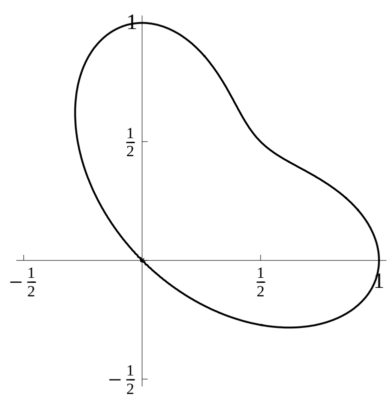
   (b) If y is an unknown function of x, express $\frac{d}{dx}(y^3)$ in terms of y and $\frac{dy}{dx}$.

2. Let's look at a circle of radius 5 centered at the origin. Find the slope of the line tangent to this circle at the point $(4, -3)$.

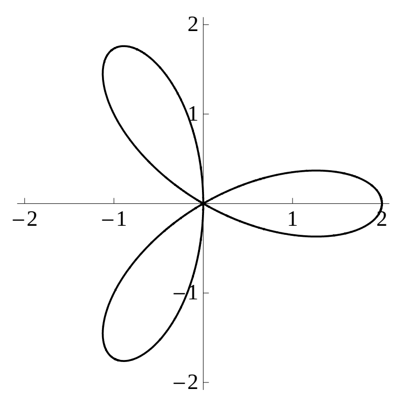
Here are some other implicit curves:



Heart curve
$(x^2 + y^2 - 1)^3 - x^2y^3 = 0$

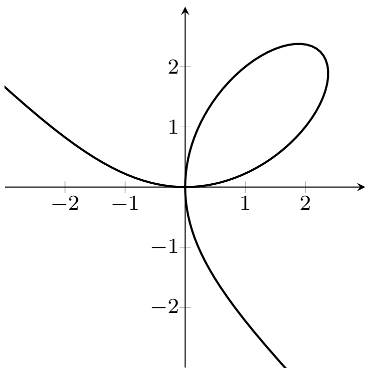


Crooked egg curve
$(x^2 + y^2)^2 = x^3 + y^3$



Trifolium
$(x^2 + y^2)^2 = 2(x^3 - 3xy^2)$

3. Find the equation for the tangent line to $x^3 + y^3 = \frac{9}{2}xy$, the *folium of Descartes*, at the point $(1, 2)$.

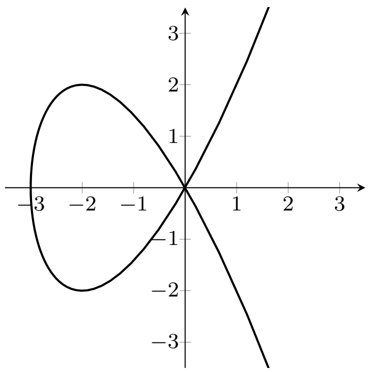


4. Let's think again about the circle in #2. Imagine that a bug is crawling along this circle. At a certain instant, the bug is at the point $(4, -3)$, and its y-coordinate is decreasing at an instantaneous rate of 2 units per second.

   (a) At this instant, is the bug going clockwise or counterclockwise?

   (b) At this instant, how fast is the bug's x-coordinate decreasing?

5. At what point(s) does the tangent line to $y^2 = x^3 + 3x^2$ have slope 0?



6. Find the line tangent to the heart curve $(x^2 + y^2 - 1)^3 - x^2y^3 = 0$ at $(-1, 1)$.

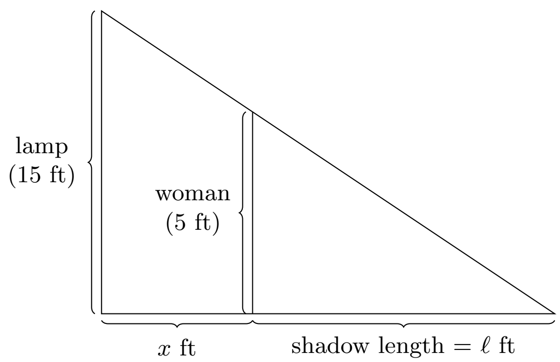
37 Related Rates

1. Thabo is working on the following problem:

   It is evening and a 5-foot-tall woman is standing by a 15-foot-high street lamp on a cobbled road, waiting for her ride home from work. When she sees her wife pull her car up to the curb she begins to walk away from the light at a rate of 4 feet per second. How fast is the length of her shadow changing?

   (a) First, before diving straight in, Thabo wants to make a reasonable estimation of what his result will be. Will the woman's shadow be increasing or decreasing in length for the above scenario? (You can try to run an experiment just like it with a light source to check your answer!)

   (b) Thabo has drawn the following picture and labeled some variables but is unsure what to do next.



   i. Thabo knows it's always a good idea to start by figuring out your goal and what you know, so he's trying to do that. Which of the following best describes the problem Thabo is trying to solve in terms of time t and the variables x, ℓ in the picture?

   A. Goal: Find $\frac{d\ell}{dx}$. What we know: $x = 4$.

   B. Goal: Find $\frac{d\ell}{dt}$. What we know: $x = 4$.

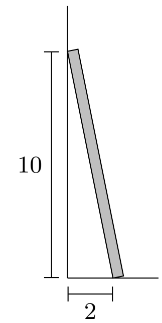
   C. Goal: Find $\frac{dx}{dt}$. What we know: $\frac{d\ell}{dt} = 4$.

   D. Goal: Find $\frac{d\ell}{dt}$. What we know: $\frac{dx}{dt} = 4$.

   (c) Write down an equation relating x and ℓ. (*Hint:* Use similar triangles!)

   (d) Solve Thabo's problem.

2. Your calculus book is sitting on your desk, leaning against the wall as shown. The book is slowly sliding down the wall, with the top of the book sliding down at a rate of 1 inch per second. We'd like to find the (instantaneous) rate at which the bottom of the book is sliding at the moment shown.



   (a) *Thought question.* Do you think the rate we're looking for is greater than, equal to, or less than 1 inch per second?

   Now, we'll find the rate exactly.

   (b) The picture drawn above is a "snapshot" showing the book at a particular moment in time. Draw a "generic" picture which could apply at any time. On this picture, give variable names to quantities that are changing over time, and label any values that are constant.

   (c) What's the goal in this problem? (What rate are you looking for?)

   (d) What rates are given in the problem?

   (e) What relationships can you come up with among your variables? Write an equation that relates your variables.

   (f) Use implicit differentiation on your equation from (e) to solve for the rate you're looking for.

   (g) If the top of the book continues to slide at a constant rate of 1 inch per second, does the bottom of the book slide at a constant rate? If not, how does this rate change? Explain your answer using a formula.

3. Two cars are approaching an intersection. A red car, approaching from the north, is traveling 20 feet per second and is currently 60 feet from the intersection. A blue car, approaching from the west, is traveling 30 feet per second and is currently 80 feet from the intersection. At this moment, is the distance between the two cars increasing or decreasing? How quickly?

4. An oil tank in the shape of an inverted cone has height 10 m and radius 6 m. When the oil is 5 m deep, the tank is leaking oil from the tip at a rate of 2 m^3 per day. How quickly is the height of the oil in the tank decreasing at this moment?

   Note: The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.

5. At noon, you are dashing through Harvard Yard to get to class and notice a friend 100 feet west of you, also running to class. If you are running south at a constant rate of 450 ft/min (approximately 5 mph) and your friend is running north at a constant rate of 350 ft/min (approximately 4 mph), how fast is the distance between you and your friend changing at 12:02 pm?

38 Unit Circle Trigaonometry

1. Sketch a graph showing the total hours of daylight each day for...

   (a) One week starting Dec. 22, 2016.

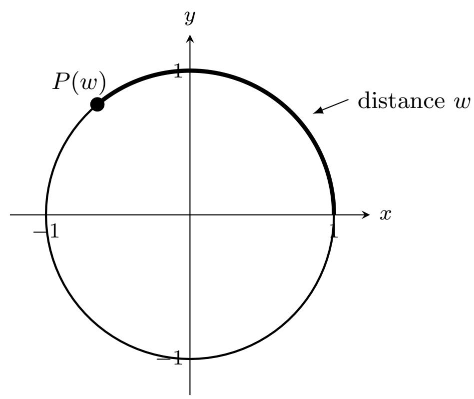
   (b) Three months starting Dec. 22, 2016.

   (c) Twelve months starting Dec. 22, 2016.

   (d) Five years starting Dec. 22, 2016.

Definition of sine and cosine. If w is a real number, we'll define $\sin w$ ("sine of w") and $\cos w$ ("cosine of w") using the following procedure. Start at the point $(1, 0)$ on a circle of radius 1 centered at the origin (we call this the unit circle).

If $w \geq 0$, travel along the circle in a counterclockwise direction a distance w units; if $w < 0$, travel along the circle in a clockwise direction a distance $|w|$ units. Call the point you've arrived at $P(w)$.



We'll define $\cos(w)$ to be the x-coordinate of $P(w)$ and $\sin(w)$ to be the y-coordinate of $P(w)$. That is, $P(w)$ has coordinates $(\cos(w), \sin(w))$.

2. When you travel halfway around the unit circle, you've gone a little more than 3.1 units. Is this surprising? Why or why not? What does this have to do with the formula for the circumference of a circle?

3. Use the calibrated unit circle to approximate the following quantities. When you're able to find an exact answer, use an "=" sign; when you are approximating, use "≈".

(a) $\sin 1$

(b) $\cos 1$

(c) $\sin(2)$

(d) $\sin(-2)$

(e) $\cos(3)$

(f) $\cos(-3)$

(g) $\sin(.5)$

(h) $\sin(.5 + 2\pi)$

(i) $\sin(.5 + \pi)$

(j) $\sin(\pi - 0.5)$

(k) $\cos(-\pi)$

(l) $\sin \frac{3\pi}{2}$

(m) $\cos\left(-\frac{11\pi}{2}\right)$

(n) $\sin \pi$

(o) $\sin 8$

4. Use the unit circle to answer the following questions.

   (a) If t is a positive number, what is the relationship between $\sin t$ and $\sin(-t)$? What does this tell you about the graph of the function $\sin t$?

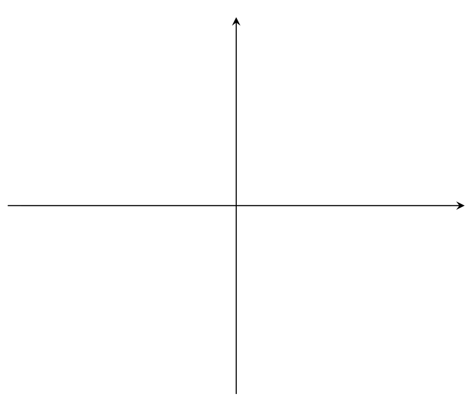
   (b) If t is a positive number, what is the relationship between $\cos t$ and $\cos(-t)$? What does this tell you about the graph of the function $\cos t$?

   (c) If t is a positive number, what is the relationship between $\sin t$ and $\sin(t + 2\pi)$? Does the same hold for cosine? What does this tell you about the graphs of $\sin t$ and $\cos t$?

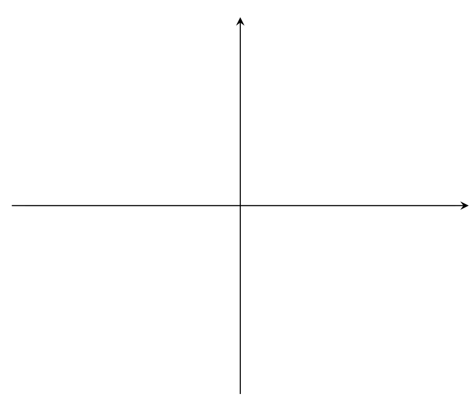
   (d) If t is a positive number, what is the relationship between $\sin t$ and $\sin(t + \pi)$?

5. Using the unit circle, sketch the graphs of $\sin t$ and $\cos t$. Make sure your graphs match the properties you've observed so far.

Graph of $\sin t$



Graph of $\cos t$



Where are the roots of $\sin t$? That is, for what values of t is $\sin t = 0$? Where are the roots of $\cos t$?

6. Suppose A is a number in $\left[0, \dfrac{\pi}{2}\right]$ such that $\sin A = \dfrac{1}{4}$.

   (a) Find all numbers t in $[0, 2\pi]$ such that $\sin t = \dfrac{1}{4}$. (Express your answers in terms of A.)

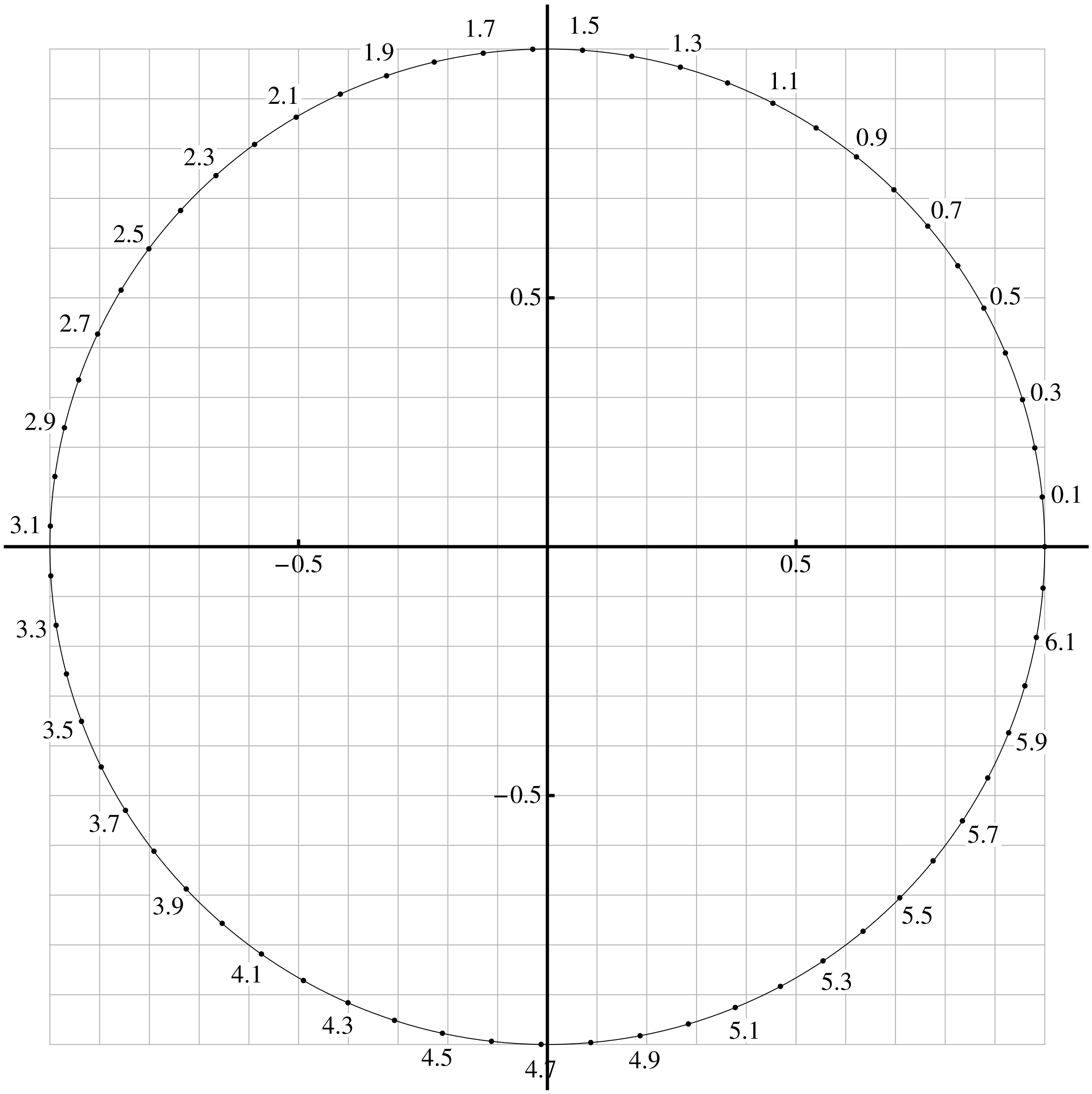
   Can you describe *all* numbers t such that $\sin t = \dfrac{1}{4}$?

   (b) Find all numbers t in $[0, 2\pi]$ such that $\sin t = -\dfrac{1}{4}$. (Express your answers in terms of A.)

7. If q is a number such that $\sin q = \frac{3}{5}$, what can you say about $\cos q$? Be as detailed and precise as possible.

8. Looking at the unit circle and your graphs, how are $\sin t$ and $\cos t$ related? Brainstorm as many different relationships between these two functions as you can.

Here is a calibrated unit circle. The values marked off on the circle represent the arc length from the point $(1, 0)$, measured counter-clockwise. The grid squares are 0.1 units on each side.

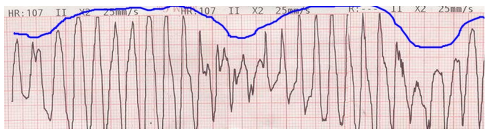


39 Periodic and Sinusoidal Functions

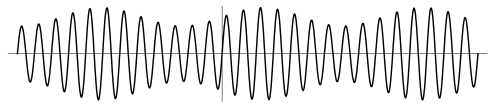
1. Informally, a periodic graph is a graph that repeats itself. Which of the following phenomena would you model using a periodic function? Which of these is not periodic?

   (a) A normal EKG: 

   (b) Torsades de pointes is a rare arrythmia of the heart which can lead to sudden death if untreated.

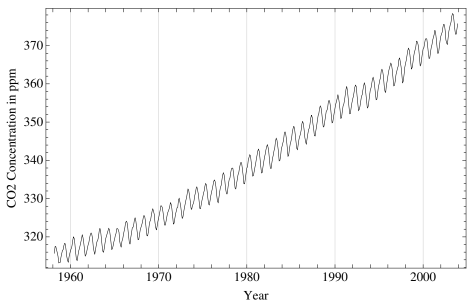
   

   Actual EKG

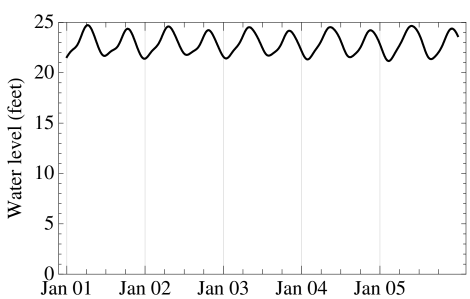
   

   Model of Torsades de pointes

   (c) At the Mauna Loa observatory in Hawaii, the amount of carbon dioxide in the atmosphere has been measured every month since 1958. Here is a graph of the data.

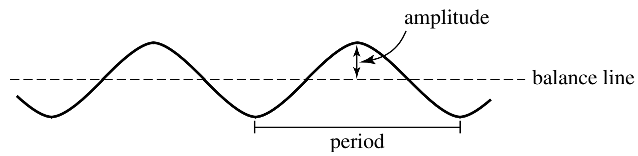
   (d) In Fall River, MA, the water level is measured every 6 minutes. Here is a graph of the data from January 1, 2012 to January 6, 2012.

2. Suppose a function f is periodic with period k. How can we mathematically express the fact that the function "repeats every k units?"

3. The term sinusoidal function or simply sinusoid is used to refer to the sine and cosine functions, as well as modifications of them obtained by shifting, flipping, stretching, and shrinking. Which of the phenomena in #1 could reasonably be modeled by a sinusoid?

Terminology for sinusoids. Period, amplitude, and balance value:



4. **Sinusoids.** Graph the following, and label the y-intercept and coordinates of a local minimum and local maximum. What are the period, amplitude, and balance value of each?

   (a) $y = \sin\left(\frac{1}{2}x\right)$

   (b) $y = 4\cos(-3x) - 5$

   (c) $y = -2\sin\left(x - \frac{\pi}{4}\right)$

Summary. For a sinusoid of the form $y = A\sin(Bx) + C$ or $y = A\cos(Bx) + C$:

Period: ___________ Amplitude: ___________ Balance value: ___________

5. A retun to last class: Model daylight for 1 year. What information would you want to know? What does the function look like?

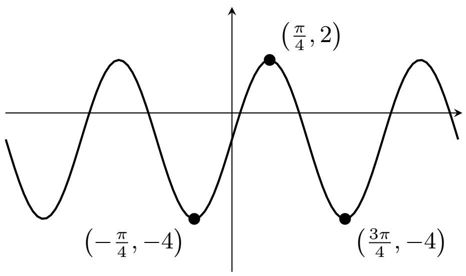
6. The most popular paid tourist attraction in the U.K. is the London Eye, a giant Ferris wheel located on the bank of the River Thames in London. The entire structure is 135 meters tall, and the wheel has a diameter of 120 meters. The Ferris wheel makes one revolution every 30 minutes. Suppose that, at time $t = 0$, the ride starts and you are in a car at the bottom of the wheel.

   (a) Let $h(t)$ be your height (in meters) above the ground at time t. Graph $h(t)$ from time $t = 0$ to time $t = 90$ minutes. (*Note:* At the highest point, you are 135 meters above the ground.)

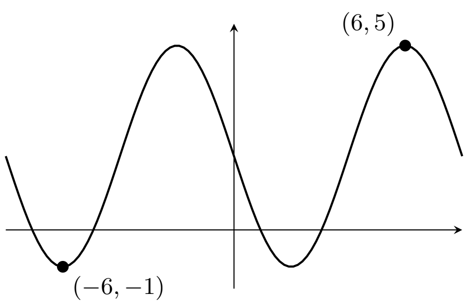
   (b) Write an equation for $h(t)$.

7. For each of the sinusoids below, determine the balance value, amplitude, and period. Then, write an equation for the sinusoid.

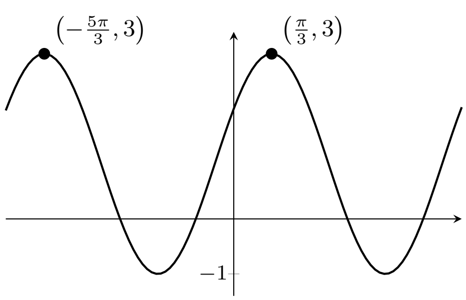
(a)



(b)



(c)



8. Write a function that models the number of hours of sunlight in Boston t years after December 22$^{\text{nd}}$ 2018.

40 Tangent Arc Length and Angles

1. (a) Using the unit circle definition of the tangent function, sketch the graph of $\tan x$.

   (b) Express $\tan x$ in terms of $\sin x$ and $\cos x$.

   (c) What are the vertical asymptotes of $\tan x$? Does this agree with your knowledge of $\sin x$ and $\cos x$? Explain.

   (d) What are the zeros of $\tan x$? Does this agree with your knowledge of $\sin x$ and $\cos x$? Explain.

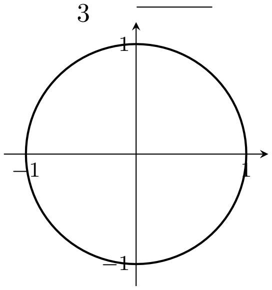
   (e) What is the period of $\tan x$?

   (f) Evaluate the following limits.

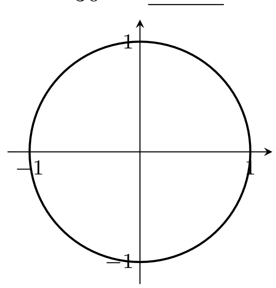
      i. $\lim\limits_{x \to (\pi/2)^-} \tan x$ ii. $\lim\limits_{x \to (\pi/2)^+} \tan x$ iii. $\lim\limits_{x \to \infty} \tan x$

2. For each of the following angles, sketch the angle in standard position. Then, convert the angle from radians to degrees, or vice versa.

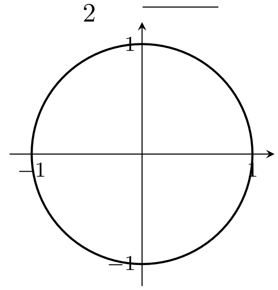
$\frac{2\pi}{3} =$ ______$^\circ$



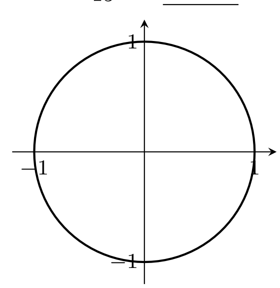
$30^\circ =$ ______



$\frac{5\pi}{2} =$ ______$^\circ$

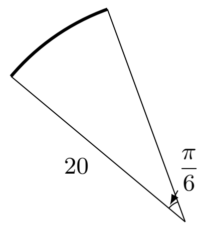


$-45^\circ =$ ______

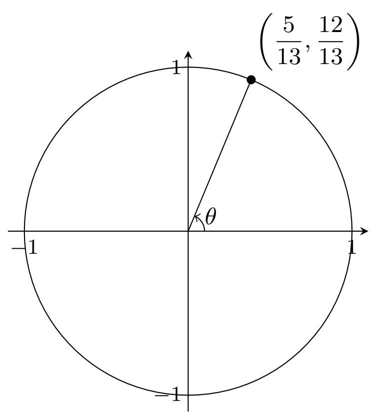


π radians $= 180^\circ$. 1 radian = _____ degrees. $1^\circ =$ _____ radians.

3. The picture shows a "circular sector," which you can think of as a wedge of a circle. What is the length of the bold piece? (The technical way to describe this bold piece is "an arc on a circle of radius 20 that subtends an angle of $\frac{\pi}{6}$.")



4. If θ is the angle shown in the picture, evaluate each of the following. (It will help to draw the angle in question on the unit circle.)



(a) $\sin\theta$

(b) $\cos(\pi - \theta)$

(c) $\sin(2\pi - \theta)$

(d) $\cos(2\pi + \theta)$

(e) $\tan(\pi - \theta)$

(f) $\sin(\pi + \theta)$

(g) How many angles A are there in $[\pi, 2\pi]$ such that $\cos A = \frac{5}{13}$? Express them in terms of θ.

5. A crop receives water by a pivot irrigation system with arm length 200m.

   If the pipe rotates only $\frac{5}{4}\pi$ radians, what is the area of the field wich is irrigated?

6. A *nautical mile* is the distance along the surface of the earth that subtends an angle of $\dfrac{1}{60}^{\circ}$.

   (a) Draw a picture to illustrate a nautical mile. What is $\dfrac{1}{60}^{\circ}$ in radians?

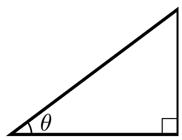
   (b) The radius of the earth is about 3960 miles. Use this to approximate a nautical mile, giving your answer in **feet**. (One mile = 5280 feet.)

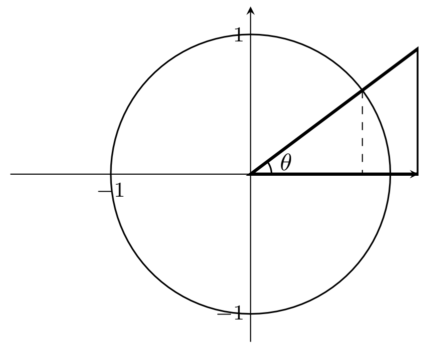
7. Make a rough sketch of $f(x) = e^{0.05x} \sin x$. What are $\lim\limits_{x \to -\infty} e^{0.05x} \sin x$ and $\lim\limits_{x \to \infty} e^{0.05x} \sin x$?

41 Right Triangle Trigonometry

Trigonometry means "measuring triangles," something people have been doing for centuries in astronomy, navigation, surveying, optics and other areas.

To see the connection between a right triangle and the unit circle, we imagine drawing the triangle together with the unit circle like this:





Let w be the arc length from (1,0) to the intersection of the circle and the hypotenuse. Use properties of arc length and angles to relate θ to w.

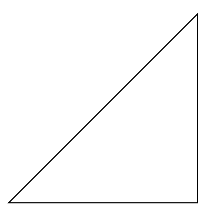
In terms of **right** triangles: $\sin\theta =$ _________ $\cos\theta =$ _________ $\tan\theta =$ _________

We also define:

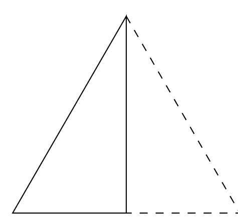
$$\text{“cosecant of } \theta\text{”} : \csc\theta = \frac{1}{\sin\theta} \qquad \text{“secant of } \theta\text{”} : \sec\theta = \frac{1}{\cos\theta} \qquad \text{“cotangent of } \theta\text{”} : \cot\theta = \frac{\cos\theta}{\sin\theta}$$

1. Here are two special right triangles, each shown with base length 1:

a right isosceles triangle



half of an equilateral triangle



   (a) Find and label the lengths of all other sides of the triangles, as well as the size of the angles in each triangle.

   (b) From your completed pictures, you can determine sine, cosine, and tangent of the acute angles in each triangle.

2. *Combining the right triangle and unit circle perspectives.*

Suppose we'd like to find $\cos\left(\frac{2\pi}{3}\right)$. We can't draw a right triangle with an angle of $\frac{2\pi}{3}$ (why not?), so we'll combine our knowledge of the unit circle and right triangles.

(a) Sketch the unit circle and the angle $\frac{2\pi}{3}$ in standard position. (The most important thing to get right in your sketch is the quadrant that the terminal side of the angle is in.)

(b) On your sketch from (a), draw a perpendicular from the point $P\left(\frac{2\pi}{3}\right)$ to the x-axis. You should now see a triangle in your picture. We call this the reference triangle for the angle $\frac{2\pi}{3}$.

(c) What is the angle in the reference triangle between the hypotenuse and the x-axis? This is called the reference angle for $\frac{2\pi}{3}$.

(d) Using your reference triangle, you should now be able to see what $\cos\left(\frac{2\pi}{3}\right)$ is. What is it? How about $\sin\left(\frac{2\pi}{3}\right)$?

3. You're interested in knowing the height of a tall tree. You position yourself so that your line of sight to the top of the tree makes a $60°$ angle with the horizontal. You measure the distance from where you stand to the base of the tree to be 45 feet. How tall is the tree, if your eyes are five feet above the ground?

4. You are standing on an overpass, 35 feet above the street, waving to a friend who is at the window of a high-rise dormitory across the overpass. The angle of depression to the bottom of the dormitory (on street level) is $15°$. The angle of elevation to your friend's window is $45°$. What is your friend's elevation?

5. Use the same idea as in #2 to find **exact** values for the following:

   (a) $\sin\left(\frac{4\pi}{3}\right)$ (b) $\tan\left(-\frac{19\pi}{6}\right)$

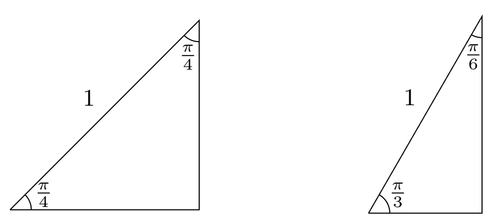
6. Give exact values for the following.

   (a) $\cot\left(-\frac{3\pi}{2}\right)$ (b) $\sec\left(\frac{13\pi}{6}\right)$

7. (a) For what values of x is $\tan x = \frac{1}{\sqrt{3}}$? Find all, and illustrate your solution using the graph of $\tan x$.

   (b) For what values of x is $\tan x = -\frac{1}{\sqrt{3}}$? Find all.

8. It's often useful to think about the two special right triangles as having hypotenuse 1 rather than base length 1. Find and label the lengths of the legs of these two triangles:



42 Derivatives of Trigonometric Functions

1. Sketch the graph of $\sin x$. Based on this, sketch the graph of the derivative of $\sin x$. From your graph, do you have a guess as to what the derivative of $\sin x$ might be?

   Q: How tall is the derivative function?

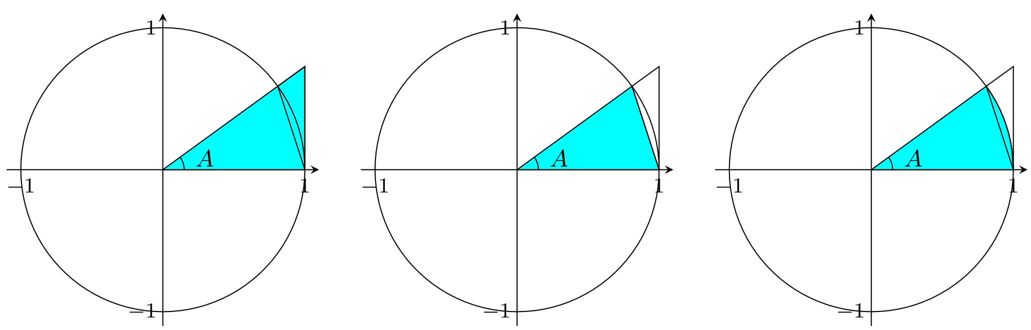
2. (a) What is the definition of the derivative of $\sin x$?

   (b) What is the definition of the derivative of $\sin x$ at $x = 0$?

3. The limit $\lim_{A\to 0} \frac{\sin A}{A}$ is an interesting one. What do you think $\lim_{A\to 0} \frac{\sin A}{A}$ is?

   This limit is not so simple, so we'll have to use some ingenuity to find it.

   (a) The following three pictures show a positive acute angle A together with the unit circle. In each picture, find the shaded area in terms of A.



(b) Put the three areas in increasing order:

$$\underline{\hspace{3cm}} \leq \underline{\hspace{3cm}} \leq \underline{\hspace{3cm}}$$

Now, multiply by 2 and divide by $\sin A$ (which is positive because $A > 0$).

$$\underline{\hspace{3cm}} \leq \underline{\hspace{3cm}} \leq \underline{\hspace{3cm}}$$

What happens as $A \to 0^+$?

(c) What does that tell you about $\lim_{A \to 0} \frac{\sin A}{A}$?

(d) In this argument, it was important that A was measured in radians. Which step would be wrong if we measured A in degrees?

There's another fact we need to find the derivative of $\sin x$:

Fact. $\sin(A + B) = \sin A \cos B + \sin B \cos A$. (You don't need to remember this formula, but you should definitely remember that $\sin(A + B) \neq \sin A + \sin B$!)

We've already seen in that the derivative of $\sin(x)$ is $\cos(x)$#2(a) that What is the definition of the derivative of $\cos x$?

What is the definition of the derivative of $\cos x$ at $x = 0$?

Can you evaluate the derivative just by thinking about the graph of $\cos x$?

4. What is the derivative of $\cos x$?

5. Find the derivative of $\tan x$. *Do* simplify your answer.

Derivatives of the six basic trigonometric functions. (You should memorize these.)

$\frac{d}{dx}(\sin x) =$ ________________ $\frac{d}{dx}(\cos x) =$ ________________ $\frac{d}{dx}(\tan x) =$ ________________

$\frac{d}{dx}(\csc x) = -\csc x \cot x$ $\frac{d}{dx}(\sec x) = \sec x \tan x$ $\frac{d}{dx}(\cot x) = -\csc^2 x$

(You'll show the last three on your homework.)

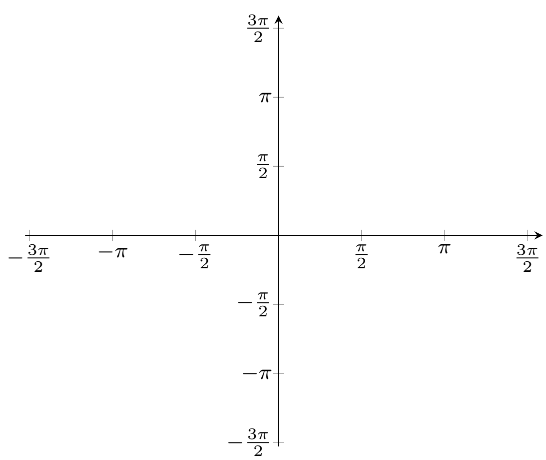
43 Inverse Trigonometric Functions

1. **Warm Up.** Think of two different values of θ such that $\sin\theta = \frac{1}{2}$.

2. A New York lawyer is working on the case of an elderly woman who twisted her ankle while chasing a runaway shopping cart down a ramp outside a Manhattan grocery store. The curb is 6.5 inches high and the flat ground from the edge of the curb to the end of the ramp measures 50 inches. New York City law specifies that ramps must have a sloping angle of no more than 5 degrees. He needs to know whether the construction of this ramp was in accordance with the law. What equation must we solve in order to find the sloping angle of the ramp?

3. *Defining arcsin.*

   (a) On the axes below, sketch $y = \sin x$, and state the domain and range of the function $\sin x$.



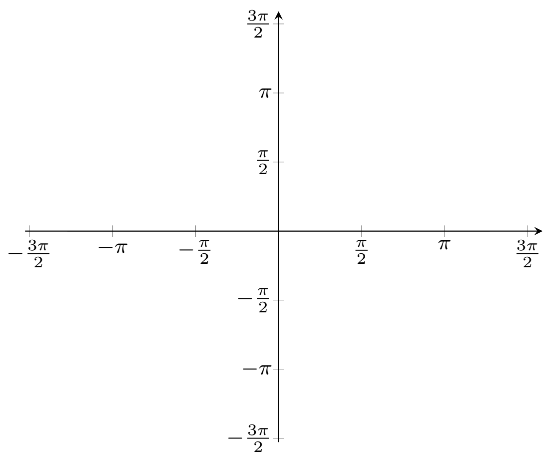
   Domain of $\sin x$: ______________

   Range of $\sin x$: ______________

   (b) We would *like* to define $\arcsin x$ to be the inverse function of $\sin x$. Why can't we do that?

   (c) Can we restrict $\sin x$ to an interval on which it is invertible?

(d) If we restrict $\sin x$ to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then it is invertible; we'll define $\arcsin x$ to be the inverse of this restricted function. On the axes below, sketch $y = \arcsin x$. What are the domain and range of the function $\arcsin x$?



Domain of $\arcsin x$: ______________

Range of $\arcsin x$: ______________

(e) If $P = \arcsin Q$, how could you express Q in terms of P? ______________

In words, $\arcsin Q$ is ____.

(f) What is $\arcsin\left(\frac{1}{2}\right)$?

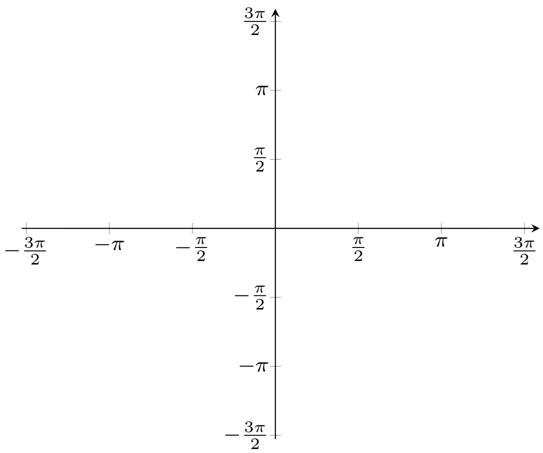
⚠ The function $\arcsin x$ is also written as $\sin^{-1} x$. It's important to remember that $\sin^{-1} x$ refers to $\arcsin x$, *not* to $(\sin x)^{-1}$; $(\sin x)^{-1}$ is the same as $\frac{1}{\sin x}$, which is the same as $\csc x$. (Yes, this notation is inconsistent: $\sin^2 x$ *is* the same thing as $(\sin x)^2$. This is one of those quirks that you just have to remember!)

(g) True or false: since $\sin\left(\frac{11\pi}{6}\right) = -\frac{1}{2}$, $\sin^{-1}\left(-\frac{1}{2}\right) = \frac{11\pi}{6}$.

4. Similarly, we'll define:

   - $\arccos x =$
   - $\arctan x =$

   Graph $\arctan x$. What are its domain and range?



Domain of $\arctan x$: ______________

Range of $\arctan x$: ______________

Although we call $\arcsin x$, $\arccos x$, and $\arctan x$ *inverse trigonometric functions*, remember that they are not really the inverse functions of $\sin x$, $\cos x$, and $\tan x$ (because $\sin x$, $\cos x$, and $\tan x$ are not invertible!). They are only the inverse functions of $\sin x$, $\cos x$, and $\tan x$ *when each function's domain is restricted in a certain way.*

5. Is each of the following quantities defined or undefined? If it is defined, what is its value?

   (a) $\arcsin\left(\sin\frac{3\pi}{2}\right)$

   (b) $\sin\left(\arcsin\frac{3\pi}{2}\right)$

6. *Using inverse trig functions to solve equations.*

   (a) Find all values of x in $[0, 2\pi]$ such that $\sin x = \frac{1}{2}$; give exact answers.

   (b) Find all values of x in $[0, 2\pi]$ such that $\sin x = \frac{1}{3}$; give exact answers (which will involve inverse trigonometric functions).

   (c) Solve $\cos x = -0.7$ on $[0, 2\pi]$.

   (d) Find all values of x such that $\cos x = -0.7$.

44 Solving Trigonometric Equations

1. Over the next week, the tides at Pleasant Bay on Cape Cod will be roughly periodic, with a period of 12.5 hours and a 3.8-foot difference between high and low tides. At 6 am this morning, the water was at "mean sea level" (half-way between high and low tide), and the water level was falling.

   (a) Which of the following best models the height of the water, relative to sea level, t hours after 6 am?

   A. $1.9\sin\left(\dfrac{2\pi}{12.5}t\right)$ B. $-1.9\sin\left(\dfrac{2\pi}{12.5}t\right)$ C. $1.9\cos\left(\dfrac{2\pi}{12.5}t\right)$ D. $-1.9\cos\left(\dfrac{2\pi}{12.5}t\right)$

   (b) There is a causeway at Pleasant Bay that becomes impassable when the water is 1.7 feet above sea level. If we want to know the first time after 6 am that the causeway will become impassable, what equation do we need to solve?

2. *Solving* $\sin x = a$, $\cos x = a$, *and* $\tan x = a$. Solve each of the following equations.

   (a) Solve $\cos x = -\frac{1}{2}$ on $[-\pi, 3\pi]$.

   (b) Solve $\cos u = 0.4$ on $[0, 3\pi]$.

   (c) Find all solutions of $\tan u = -3$.

   (d) Solve $\sin t = -0.3$ on $[0, 4\pi]$.

3. *Substituting for the inside of a trigonometric function.* Solve the following equations on the interval specified. (Your answers to #2 may be helpful.)

   (a) $\cos\left(\frac{x}{3}\right) = -\frac{1}{2}$ on $[-3\pi, 9\pi]$.

   (b) $5\cos\left(\frac{x}{2}\right) = 2$ on $(-\infty, \infty)$

   (c) $2\sin(3t) = \sqrt{3}$ for t in $[-\pi, \pi]$

4. *Substituting for an entire trigonometric expression.*

   $2\cos^2 x + 3\cos x + 1 = 0$ on $[-\pi, \pi]$

5. Solve.

   (a) $\sin(x^2) = -\frac{1}{\sqrt{2}}$ on $[-2, 2]$

   (b) $\sin\theta\tan\theta = -\sin\theta$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

   (c) $3^{4\sin^2 x} = 9$ for x in $[0, 2\pi]$

6. Solve the equation you found in #1(b) to find the first time after 6 am that the causeway at Pleasant Bay becomes impassable.

45 Problem Solving with Trigonometry

1. (a) Sketch the unit circle together with the angle $\sin^{-1}\left(-\frac{3}{5}\right)$ in standard position. Draw in the reference triangle for this angle.

   (b) What is $\cos\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)$?

2. Simplify each of the following to an expression that does not involve trigonometric functions or inverse trigonometric functions.

   (a) $\sin\left(\cos^{-1}\left(-\dfrac{2}{3}\right)\right)$

   (b) $\sin\left(\arctan(-2)\right)$

   (c) $\tan\left(\arctan 10\right)$

   (d) $\arctan\left(\tan 3\pi\right)$

3. In this problem, you'll graph $f(x) = \cos(\arccos x)$ and $g(x) = \arccos(\cos x)$.

   (a) What is the domain of f? How about g?

   (b) For what values of x is it true that $f(x) = x$? How about g?

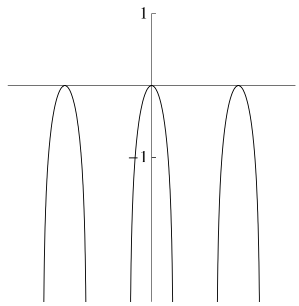
   (c) Is f periodic? If so, how often does it repeat? How about g?

   (d) If x is in $[\pi, 2\pi]$, what is $\arccos(\cos x)$?

   (e) Graph f and g. Make sure the domain and range of each graph is clear.

4. For each of the following functions, find the picture below which shows its graph.

   (a) $\arcsin(\sin x)$ (b) $\cos(x^2)$ (c) $\ln(\cos x)$ (d) $x\cos x$



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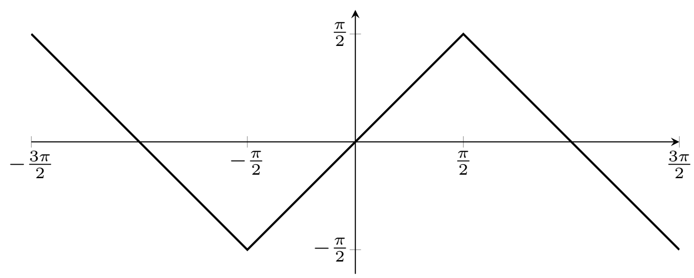

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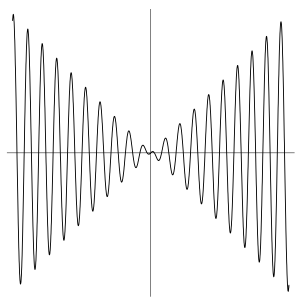

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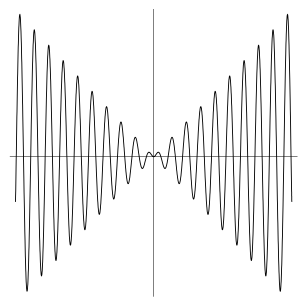

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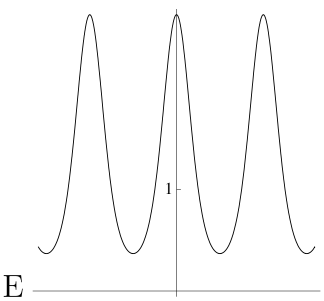

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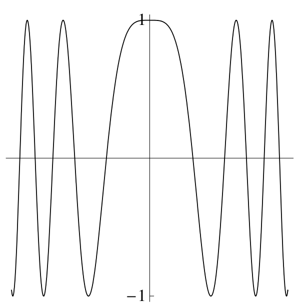

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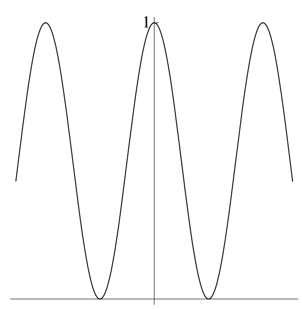

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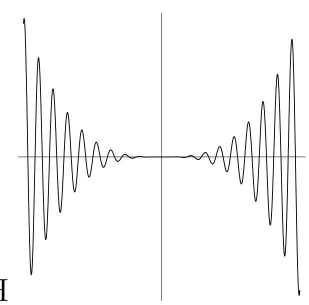


H

46 Derivatives of Inverse Trig Functions

1. (a) Sketch $y = \arctan x$, and label any horizontal and vertical asymptotes on your graph. Then, sketch a rough graph of the derivative of $\arctan x$.

Now, we'll find the derivative of $y = \arctan x$.

(b) Rewrite the equation $y = \arctan x$ as an equation involving tan rather than arctan.

(c) Use implicit differentiation on the equation you just wrote down to find $\dfrac{dy}{dx}$. (Does it match the graph you sketched?)

(d) Use simplification methods from Inverse Trig. to rewrite the denominator.

2. Sketch $\arcsin x$ and $\frac{d}{dx}(\arcsin x)$, and then use the idea of #1 to find $\frac{d}{dx}(\arcsin x)$.

Derivatives of the inverse trigonometric functions. (You should memorize these.)

$\dfrac{d}{dx}(\arcsin x) =$ ____________ $\qquad \dfrac{d}{dx}(\arccos x) = -\dfrac{1}{\sqrt{1-x^2}} \qquad \dfrac{d}{dx}(\arctan x) =$ ____________

(You'll do $\arccos x$ on your homework.)

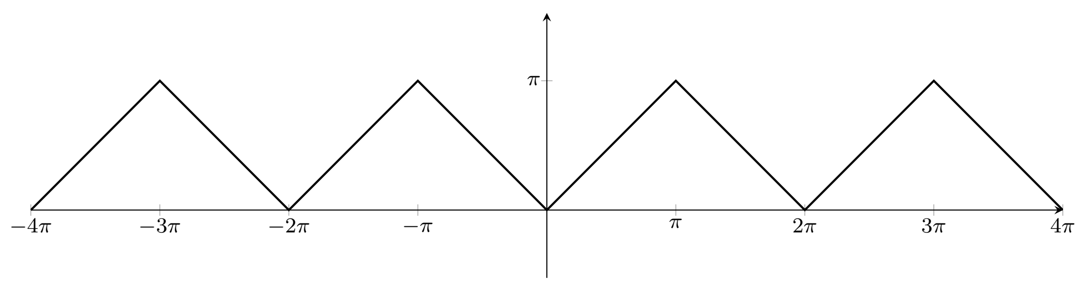
3. Differentiate the following functions. (You should *not* need implicit differentiation here; rather, use what you already know about the derivatives of $\arcsin x$, $\arccos x$, and $\arctan x$.)

   (a) $f(x) = \arcsin\left(\dfrac{1}{x}\right)$

   (b) $g(x) = e^{3x}\arccos(2x)$

   (c) $h(x) = \dfrac{\sqrt{\arctan(4x)}}{e^5}$

4. In #3 on the worksheet "Problem Solving with Trigonometry", you found that the graph of $\arccos(\cos x)$ looks like this:



   What is the derivative of $\arccos(\cos x)$? Sketch a graph of this derivative.

5. Let $f(x) = x - \arctan(2x)$.

   (a) Find all local maxima and local minima of f (give both coordinates), and determine the intervals on which f is increasing and decreasing.

   (b) Find all inflection points of f, and determine the intervals on which f is concave up and concave down.

   (c) What is $\lim_{x\to\infty} f(x)$? $\lim_{x\to-\infty} f(x)$?

   (d) What is $\lim_{x\to\infty} f'(x)$? $\lim_{x\to-\infty} f'(x)$?

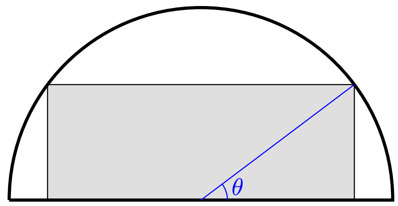
   (e) Is $f(x)$ an even function, an odd function, or neither?

   (f) Sketch the graph of $f(x)$.

   (g) Based on your work in this problem, can you determine with certainty how many roots f has?

47 Problem Solving with Inverse Trig Functions

1. Find the area of the largest rectangle that can be inscribed in a semicircle of radius 2.



2. (a) A rocket is propelled straight upward from a launching pad. An observation site is located at the same height as the launching pad, but 600 feet away from it. At the observation site, a tracking instrument is tracking the take-off of the rocket. The stationary tracking instrument is rotating to keep the rocket in the center of its line of sight. If the angle of elevation of the tracking instrument is increasing at the rate of 0.5 radians per second when the angle of elevation is $\frac{\pi}{4}$, what is the vertical speed of the rocket at that instant?

   (b) If the rocket continues climbing at the same vertical speed, does the instrument need to speed up, slow down, or continue rotating at the same speed to keep tracking the rocket?

3. (From Cody Patterson) Cody has a laser pointer mounted on a stand that can be rotated horizontally. The stand is located 4 meters from a wall. Cody begins with the laser pointer perpindicular to the wall, and he then starts to rotate the laser pointer clockwise at a constant rate of 1/6 radians per second.

   (a) When the laser dot on the wall is exactly 5 meters away from the laser pointer, how quickly is the light on the wall moving?

   (b) At what point along the wall is the laser dot moving most slowly? (You may assume that the wall is infinitely long.)

48 Indeterminate Forms

1. Evaluate the following limits, using the techniques you learned in Math Ma.

   (a) $\lim_{x\to\infty} \dfrac{x^3+1}{x^2+3x}$

   (b) $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

2. The limits in #1 were "of the form $\frac{0}{0}$," or "of the form $\frac{\infty}{\infty}$." $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are examples of what we call "indeterminate forms."

   (a) What do we mean when we say "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" is an indeterminate form?

   (b) Come up with a couple of examples of the type "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" that show that these are *indeterminate* forms.

3. Consider the limit $\lim_{x\to 1} \dfrac{\sin(\pi x)}{\frac{\pi}{4} - \arctan x}$. Can you evaluate it?

   (a) If you were to graph $\sin(\pi x)$ and $\frac{\pi}{4} - \arctan x$ and zoom in around $x = 1$, what would the graphs resemble? Can you approximate the graphs by something simpler near $x = 1$?

   (b) Using your approximations, what do you think $\lim_{x\to 1} \dfrac{\sin(\pi x)}{\frac{\pi}{4} - \arctan x}$ is?

l'Hôpital's Rule. If $\lim_{x\to a}\frac{f(x)}{g(x)}$ is a "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" type of limit, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$, if the latter limit exists or is $\pm\infty$. (This rule also works for one-sided limits.)

☞ Remember that L'Hôpital's Rule only applies to limits of the form $\frac{0}{0}$ and $\frac{\infty}{\infty}$!

(c) Use l'Hôpital's Rule to evaluate $\lim_{x\to 1}\frac{\sin(\pi x)}{\frac{\pi}{4}-\arctan x}$.

4. Evaluate the following limits. *Warning:* l'Hôpital's Rule does not apply to all of them!

(a) $\lim_{x\to\pi}\frac{\sin x}{(x-\pi)(x+3)}$

(b) $\lim_{x\to -3}\frac{\sin x}{(x-\pi)(x+3)}$.

What can you say about the graph of $f(x) = \frac{\sin x}{(x-\pi)(x+3)}$ around $x = \pi$ and $x = -3$?

(c) $\lim_{x\to 0}\frac{e^x - 1 - x}{x^2}$

(d) $\lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt[3]{x}}$

(e) $\lim\limits_{x\to 3^-} \dfrac{\ln(9-x^2)}{\arcsin(x-3)}$

(f) $\lim\limits_{x\to\infty} \dfrac{\sin x}{x}$

5. Evaluate $\lim\limits_{x\to 0^+} \sqrt{x}\ln x$. What form is this limit?

   The strategy for dealing with a limit of this form is to rewrite the product $f(x)g(x)$ as a quotient; $f(x)g(x) = \frac{f(x)}{1/g(x)}$ or $f(x)g(x) = \frac{g(x)}{1/f(x)}$. (Both equations are true; often, one will be easier to work with than the other.)

49 Relative Growth Rates

l'Hôpital's Rule. If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is a _____ or _____ type of limit, then $\lim_{x\to a} \frac{f(x)}{g(x)} =$ ____________________, if the latter limit exists or is $\pm\infty$.

1. (a) Sketch the graphs of $\ln x$ and $\sqrt[3]{x}$. Which do you think is larger when x is very large?

   (b) How could we use limits to confirm which is larger when x is very large?

   (c) Generalize: If p is any positive number, does $\ln x$ or x^p grow more quickly?

Suppose $f(x)$ and $g(x)$ are functions that are both positive when x is large. We say that $g(x)$ grows much more quickly than $f(x)$, written $g(x) \gg f(x)$, if $\lim_{x\to\infty} \frac{f(x)}{g(x)} =$ ____ (or, equivalently, if $\lim_{x\to\infty} \frac{g(x)}{f(x)} =$ ____).

2. (a) Does 1.001^x or x^2 grow more quickly? Use the limit definition of $\ll$ to support your answer.

   (b) Does 1.0001^x or $3x^3 + 7x + 5$ grow more quickly? Use the limit definition of $\ll$ to support your answer.

   (c) Generalize: If $P(x)$ is a polynomial with a positive leading coefficient[12] and $b > 0$, does b^x or $P(x)$ grow more quickly? (Does it depend on the particular polynomial, or on the value of b?)

3. All of the following functions increase without bound as $x \to \infty$. Put them in order from slowest growing to fastest.

$$1.01^x \qquad 3^{2x} \qquad 2^{3x} \qquad \sqrt[7]{x} \qquad x^{1{,}000{,}000} \qquad \log_7 x \qquad x \ln x$$

[12] The leading coefficient of a polynomial is the coefficient of the highest degree term. For instance, the leading coefficient of $-2x^4 + x^3 + 7x^2$ is -2.

4. *Using relative growth rates to reason about limits at infinity.*

   (a) Evaluate $\lim_{x\to\infty} \dfrac{2x^3+\sqrt{x}}{5x^3+4x^2+4}$.

   (b) Apply the same idea to evaluate $\lim_{x\to\infty} \dfrac{\ln x+\pi x^{0.1}+e^{-x}}{\sqrt[10]{7x}+\frac{1}{x}}$.

5. Evaluate the following limits.

   (a) $\lim_{x\to\infty} \dfrac{x^2\ln x}{3x^2+17}$ (b) $\lim_{x\to-\infty} \dfrac{e^x+x^2}{7x^3+5}$ (c) $\lim_{x\to\infty} \dfrac{x^2+1.01^x+7\sin x}{\ln x+\sqrt{x}+1.02^x}$

6. Evaluate $\lim_{x\to 0^+} \sqrt{x}e^{1/x}$. What form is this limit?

$0\cdot\infty$ limits. The strategy for dealing with a limit $\lim_{x\to a} f(x)g(x)$ of the indeterminate form $0\cdot\infty$ is to rewrite the product $f(x)g(x)$ as a quotient; $f(x)g(x) = \frac{f(x)}{1/g(x)}$ or $f(x)g(x) = \frac{g(x)}{1/f(x)}$. (Both equations are true; often, one will be easier to work with than the other.)

50 More Indeterminate Forms

1. Evaluate $\lim\limits_{x\to\infty} x\ln\left(1+\dfrac{0.06}{x}\right)$. What form is this limit?

2. Consider $\lim\limits_{h\to 0^+}(1+h)^{1/h}$.

   (a) What was your guess for the value of this limit?

   (b) The strategy for dealing with a limit of this form is to rewrite it. Rewrite $(1+h)^{1/h}$ as $e^{\text{something}}$. (What is the "something" in the exponent? Simplify it.)

   (c) As $h \to 0^+$, what happens to the "something" in the exponent of (b)? (Does it approach a limit?)

   (d) What can you conclude about $\lim\limits_{h\to 0^+}(1+h)^{1/h}$?

3. (a) Steve decides to open a bank account with an opening deposit of \$1000. Suppose that the account earns a nominal annual interest rate of 6%, compounded annually. Assuming Steve completely ignores the account after he opens it (so he doesn't make any deposits or withdrawals), how much money does the account have after 1 year? How about after t years?

   (b) Suppose that Steve had instead deposited his money in a bank that offered monthly compounding (but everything else was the same). How much money would the account have after 1 year? How about after t years?

   What is the percent growth in the account's value over one year? (This is called the annual percentage yield, or APY.)

   (c) Continuous compounding is defined to be the limit as $n \to \infty$ of having n compounding periods a year. If Steve had used a bank with continuous compounding, how much money would he have after 1 year? How about after t years?

4. Evaluate the following limits. It is up to you to decide on an appropriate method.

   (a) $\displaystyle\lim_{x\to(\pi/2)^-} \frac{\tan x}{\sec x}$

   (b) $\displaystyle\lim_{x\to 0^-} (1-2x)^{1/\cos x}$

   (c) $\displaystyle\lim_{x\to 0^-} (1-2x)^{1/\sin x}$

   (d) $\displaystyle\lim_{x\to -\infty} xe^x$

   (e) $\displaystyle\lim_{x\to(\pi/2)^-} \sec 7x \cos 3x$

   (f) $\displaystyle\lim_{x\to \infty} (\arctan x)(\ln x)$

51 Net Change

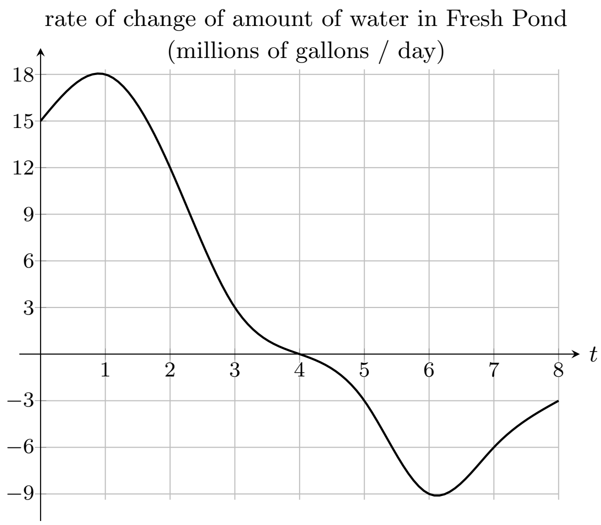
1. On Patriot's Day last year, Meb ran the Boston Marathon. For the first two hours, Meb ran at a constant speed of 7 mph. For the next two hours, Meb's speed was given by $v(t) = (t-4)^2 + 3$, where t was hours after he started the race.

   Meb's 12-year-old brother would like to use this information to figure out how far Meb ran in the first 4 hours of the race. How could he *approximate* this distance? (He's only 12, so he doesn't know any calculus!)

2. Last year, Meb's friend Andrea also ran the Boston Marathon. For the first two hours, Andrea ran together with Meb (at a constant speed of 7 mph). For the next two hours, Andrea's speed was given by $v(t) = 11 - 2t$, where t was hours after she started the race. How far did Andrea run in the first 4 hours? Is your answer an approximation, or is it exact?

3. Fresh Pond is a reservoir about 2 miles from Harvard, which holds drinking water for the city of Cambridge. The amount of water in the pond changes over time due to rainfall and water being pumped in and out of the reservoir. Suppose that the following graph shows the *rate* at which water is entering or leaving the reservoir (measured in millions of gallons per day) at time t, where t is measured in days.

   (a) How could you *approximate* the net change in the amount of water in Fresh Pond between $t = 0$ and $t = 8$?



   (b) How could you visualize the *exact* net change in the amount of water in Fresh Pond between $t = 0$ and $t = 8$?

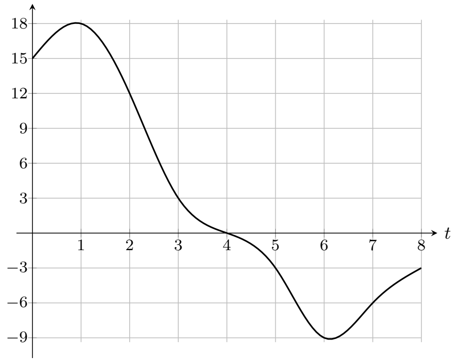
   (c) Does the pond have more water at time $t = 3$ or time $t = 4$?

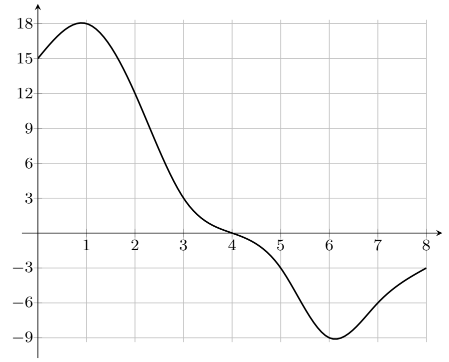
   (d) Does the pond have more water at time $t = 3$ or time $t = 6$?

   (e) Does the pond have more water at time $t = 0$ or time $t = 8$?

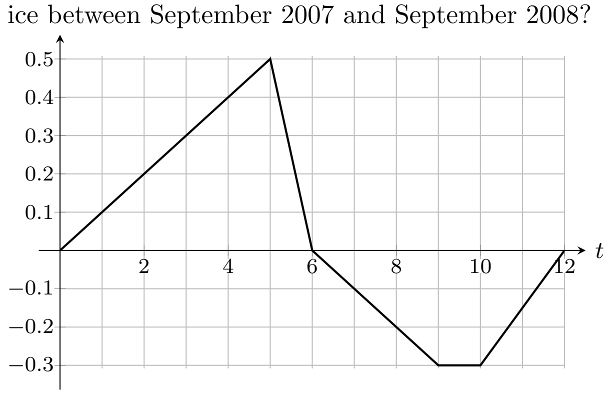
(f) Use a left-hand sum with 4 subintervals to approximate the net change in the amount of water between $t = 0$ and $t = 8$. (We often use the notation L_4 for "left-hand sum with 4 subintervals.")

Then, use a right-hand sum with 4 subintervals (we'll typically use the notation R_4 for this).





4. Arctic sea ice grows in the Arctic winter and melts in the Arctic summer. The following is a model[13] of the rate of growth and melt of Arctic Sea ice, $r(t)$, from Sept 2007 ($t = 0$) to Sept 2008 ($t = 12$). The rate is measured in millions of square kilometers per month. What is the net change in Arctic sea ice between September 2007 and September 2008?

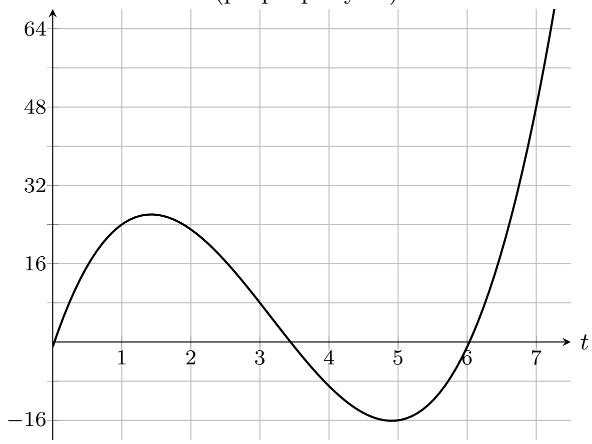


[13] While the model is hypothetical, it is loosely based on data found at the National Snow and Ice Data Center website http://nsidc.org/arcticWS21_fig1_seaicenews/2008.

52 The Definite Integral

1. The population of a tiny island country changes over time, as people immigrate and emigrate. Suppose $f(t)$ gives the rate of change of the population, where t is measured in years.

$f(t)$ = rate of change of island's population
(people per year)



   (a) Use a right-hand sum with 3 rectangles to estimate the net change in population between $t = 1$ and $t = 7$.

   (b) How would you visualize the *exact* net change in population between $t = 1$ and $t = 7$?

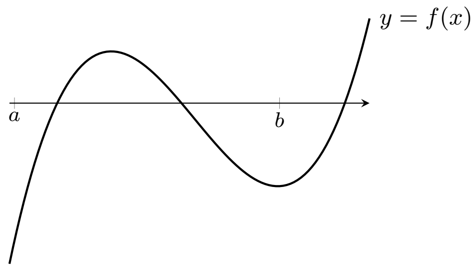
> **Notation.** We write this quantity as $\int_1^7 f(t)\,dt$. The variable t here is a "dummy variable", so we could also write it as $\int_1^7 f(x)\,dx$ (or use any other letter in place of t without changing the meaning). We read this as the definite integral of f from 1 to 7.

   (c) Was the population larger at time $t = 1$ or time $t = 7$? How do you know?

   (d) Suppose the population at time $t = 2$ was 2000 people. Write an expression involving a definite integral which gives the number of people at time $t = 4$.

The Definite Integral. We have three major ways of thinking about the definite integral $\int_a^b f(x)\,dx$:

1. If $a < b$, we visualize it as: ____



2. If f is a rate function, then we interpret $\int_a^b f(x)\,dx$ as ______________________________

3. $\int_a^b f(x)\,dx$ is actually defined to be ____________________________________

2. (a) Evaluate $\int_{-1}^{2} (|t| - 1)\,dt$.

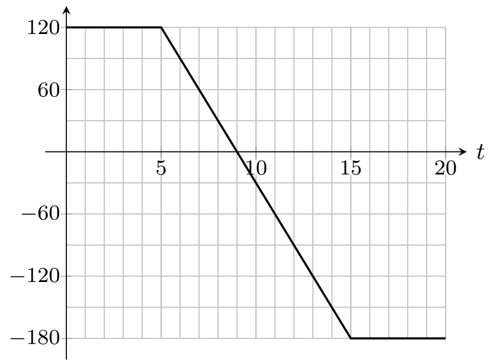
   (b) Suppose $|t| - 1$ gives the rate (in gallons per hour) of water flowing into and out of a tank t hours after noon on a particular day. Which of the following is a correct interpretation of $\int_{-1}^{2} (|t| - 1)\,dt$?

   A. It is the amount of water in the tank at 2 pm.

   B. It is the amount of water in the tank at 11 am.

   C. It is (amount of water in the tank at 2 pm) $-$ (amount of water in the tank at 11 am).

   D. It is (amount of water in the tank at 11 am) $-$ (amount of water in the tank at 2 pm).

3. 

The function $r(t)$ graphed to the left gives the rate at which water is flowing into or out of a backyard swimming pool. Time t is measured in hours, and the rate $r(t)$ is measured in gallons per hour. At time $t = 0$, there are 700 gallons of water in the pool.

(a) Write a definite integral that represents the net change in the amount of water in the pool from time $t = 0$ to $t = 6$ hours, and find this net change exactly.

(b) How much water is in the pool at time $t = 6$?

(c) When is water entering the pool at the greatest rate? What is this greatest rate?

(d) When is the amount of water in the pool greatest? At that time, how much water is in the pool?

(e) Between $t = 4$ and $t = 12$, what is the net change in the amount of water in the pool? Express your answer as a definite integral.

(f) Recall that, at $t = 0$, the pool contains 700 gallons of water. Is there ever another time T at which the pool contains 700 gallons of water? If so, approximate T on the graph.

(g) Let $W(t)$ be the amount of water in the pool at time t. Sketch a rough graph of $W(t)$, and make sure your graph agrees with your answers to the previous parts of this problem.

(h) Put the following quantities in increasing order: $\int_5^8 r(t)\,dt \quad \int_5^9 r(t)\,dt \quad \int_5^{10} r(t)\,dt \quad \int_5^{11} r(t)\,dt.$

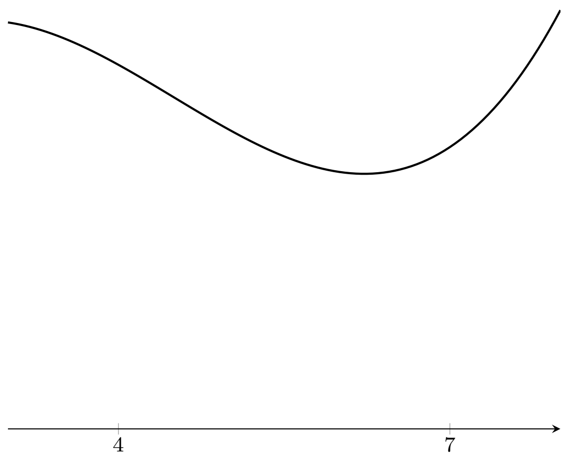
53 Definition of the Definite Integral

1. *Recap!*

   (a) So far, we have 3 main ways of thinking about the definite integral $\int_a^b f(x)\,dx$. What are they?

   (b) So far, when are we able to evaluate $\int_a^b f(x)\,dx$ exactly?

2. If f is the function shown, sketch the right-hand sum R_5 that approximates $\int_4^7 f(x)\,dx$, and write out R_5 as a sum.



3. Suppose we have a function $y = f(x)$ and want to estimate $\int_a^b f(x)\,dx$ using R_n, the right-hand sum with n rectangles.

   (a) We start by **slicing** the area into n pieces of equal width Δx (pictured in Figure 2 with $n = 6$ and in Figure 3 with $n = 12$).

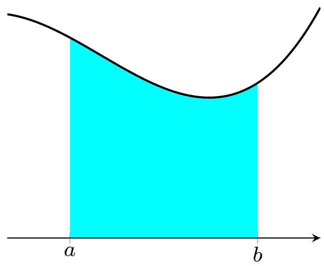


Figure 1

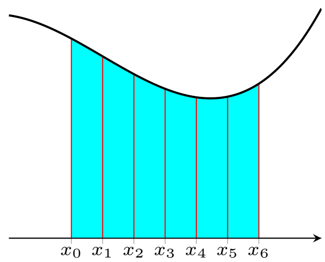


Figure 2

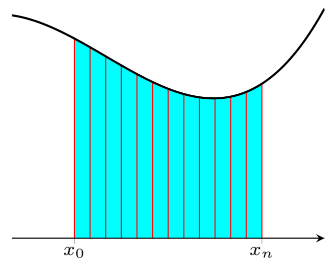


Figure 3

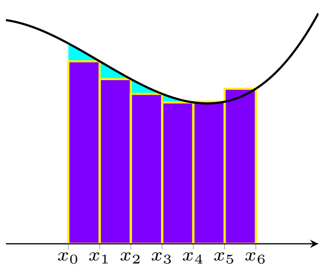
   What is Δx in terms of a, b, and n? ______________

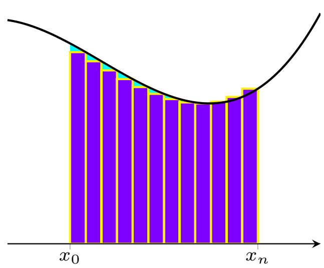
   You can see that we label the values where we slice as $x_0, x_1, \ldots, x_n$. What is x_0 (in terms of $a, b, \Delta x$)? _______

   What is x_1? ______________ x_2? ______________ x_5? ______________

   In general, what is x_k? ______________

   (b) Since we are doing a right-hand sum, we **approximate** the area of each rectangle using a rectangle whose height is the height of the right side of the slice, as shown.





   What is the signed area of the 1st rectangle? ____________________

   What is the signed area of the 4th rectangle? ____________________

   What is the signed area of the k-th rectangle? ____________________

   (c) What is R_n?

   (d) What if you instead wanted to use a left-hand sum? What would the signed area of the k-th rectangle be? What is L_n?

The official definition of the definite integral. The <u>definite integral of $f(x)$ from $x = a$ to $x = b$</u>, denoted $\int_a^b f(x)\,dx$, is defined to be

$$\lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} f(x_k)\Delta x}_{R_n} = \lim_{n\to\infty} \underbrace{\sum_{k=0}^{n-1} f(x_k)\Delta x}_{L_n}, \text{ where } \Delta x = \frac{b-a}{n} \text{ and } x_k = a + k\Delta x$$

provided these limits exist.

If f is continuous on $[a, b]$, then the limits always exist.

4. In this problem, we'll evaluate $\int_0^1 x^2\,dx$ using the definition of the definite integral as $\lim_{n\to\infty} R_n$.

   (a) Write a formula for R_n and simplify it. The following fact will be useful.

   $$1^2 + 2^2 + 3^2 + \cdots + n^2, \text{ which can also be written as } \sum_{k=1}^{n} k^2, \text{ is equal to } \frac{2n^3 + 3n^2 + n}{6}$$

   (b) What is R_4? (Simplify your answer.)

   (c) What is $\int_0^1 x^2\,dx$?

5. Let $f(x) = \sin x$. Suppose we want to approximate $\int_0^{1.5} f(x)\,dx$ using various left-hand and right-hand sums.

   (a) Put the following expressions in ascending order.

   $$\int_0^{1.5} f(x)\,dx \qquad L_4 \qquad R_4 \qquad L_{20} \qquad R_{20} \qquad L_{100} \qquad R_{100}$$

   (b) Find $|R_4 - L_4|$. How can you visualize this? (What's the least work you can do to find $|R_4 - L_4|$? Do you need to calculate R_4 and L_4?)

   (c) Find $|R_{100} - L_{100}|$.

   (d) How large do we need n to be ensure $|R_n - L_n| < 0.05$? Why might we care to have $|R_n - L_n| < 0.05$?

54 Properties of the Definite Integral

1. Remember that we have 3 main ways of thinking about the definite integral. What are they?

   For the definition of the definite integral, clearly define all notation used.

2. *Properties of the Definite Integral.*

   (a) How do $\int_1^4 f(x)\,dx$ and $\int_4^1 f(x)\,dx$ relate?

   (b) Can you simplify $\int_1^3 f(x)\,dx + \int_3^4 f(x)\,dx$? How about $\int_1^3 f(x)\,dx + \int_3^2 f(x)\,dx$?

   (c) How does $\int_a^b 3f(x)\,dx$ relate to $\int_a^b f(x)\,dx$? Why?

   (d) How does $\int_a^b [f(x)+g(x)]\,dx$ relate to $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$? Why?

Summary.

1. $\displaystyle\int_b^a f(x)\,dx =$

2. $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx =$

3. If k is a constant, $\displaystyle\int_a^b kf(x)\,dx =$

4. $\displaystyle\int_a^b [f(x) + g(x)]\,dx =$

3. Suppose you know that $\displaystyle\int_2^5 f(t)\,dt = 7$. Do you have enough information to evaluate each of the following integrals? If so, evaluate; if not, explain why not.

   (a) $\displaystyle\int_2^5 [f(t) + 4]\,dt$

   (b) $\displaystyle\int_2^5 f(t + 4)\,dt$

   (c) $\displaystyle\int_5^2 4f(t)\,dt$

   (d) $\displaystyle\int_2^4 [f(t) + t]\,dt + \int_4^5 f(t)\,dt$

4. At the Blue Bell ice cream factory, milk is stored in a 40,000 gallon tank. Let $f(t)$ be the rate (in gallons per hour) of milk flowing into and out of the tank one day at time t, where t is measured in hours after noon. At 3 pm, there are 15,000 gallons of milk in the tank.

   (a) How much milk is in the tank at 7 pm? Your answer should involve a definite integral.

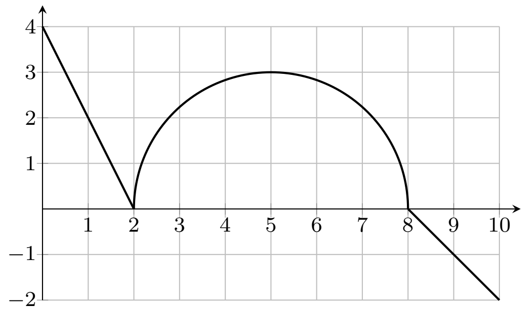
   (b) How much milk is in the tank x hours after noon?

   (c) What does ${}_0A_f(7)$ represent?

   (d) What does ${}_0A_f(x)$ represent?

   (e) Let $M(x)$ be the amount of milk in the tank x hours after noon. How would you express "the instantaneous rate at which milk is flowing into/out of the tank at 6 pm" in terms of M? In terms of f? In terms of ${}_3A_f$?

5. Here is the graph of a function f. (The graph is composed of two straight line segments and one semicircle of radius 3.)



Evaluate each of the following.

${}_0A_f(0) =$	${}_2A_f(0) =$
${}_0A_f(2) =$	${}_2A_f(2) =$
${}_0A_f(5) =$	${}_2A_f(5) =$
${}_0A_f(8) =$	${}_2A_f(8) =$
${}_0A_f(10) =$	${}_2A_f(10) =$

Do you see any relationship between ${}_0A_f(x)$ and ${}_2A_f(x)$? Can you explain this relationship?

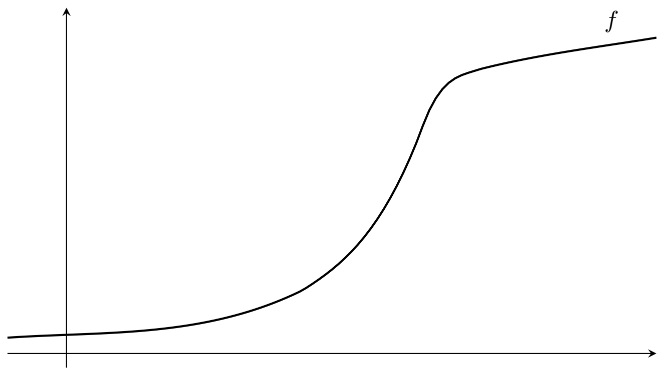
55 Fundamental Theorem of Calculus Part 1

Last time, we defined the area function ${}_aA_f(x)$ by ${}_aA_f(x) = \int_a^x f(t)\,dt$. Today, we're going to focus on the derivative of this area function.

1. What are the three ways we can think about the definite integral?

2. (a) What does the definition of the derivative say that ${}_aA_f{}'(x)$ is?

By the definition of the derivative, ${}_aA_f{}'(x) = \lim_{h\to 0} \frac{{}_aA_f(x+h) - {}_aA_f(x)}{h}$. Let's think of f as a positive, increasing function for now, so its graph might look something like this:



(b) For now, let's think of h as a positive number (are we interested in what happens when it's big or small?). On the graph above, draw an area that represents the quantity ${}_aA_f(x+h) - {}_aA_f(x)$. Label a, x, and h on your picture.

(c) We'd like to estimate ${}_aA_f(x+h) - {}_aA_f(x)$. Use a single rectangle to approximate ${}_aA_f(x+h) - {}_aA_f(x)$ in a way that guarantees an underestimate. (Try to make your approximation as accurate as possible while ensuring that it is an underestimate.)

(d) Now, use a single rectangle to approximate ${}_aA_f(x+h) - {}_aA_f(x)$ in a way that guarantees an overestimate.

(e) Use the two previous parts to write an inequality:

$$\underline{\hspace{4cm}} \le {}_aA_f(x+h) - {}_aA_f(x) \le \underline{\hspace{4cm}}$$

Use the previous line to write an inequality about the quotient $\frac{{}_aA_f(x+h) - {}_aA_f(x)}{h}$:

$$\underline{\hspace{4cm}} \le \frac{{}_aA_f(x+h) - {}_aA_f(x)}{h} \le \underline{\hspace{4cm}}$$

(f) Using what you've done so far, what can you say about the limit $\lim_{h\to 0} \frac{{}_aA_f(x+h) - {}_aA_f(x)}{h}$? Do you have enough information to evaluate it? If so, what is it? If not, why not?

3. Suppose $f(t)$ represents the rate at which water is flowing into or out of a pool at time t, where t is measured in hours after noon. At 2 pm, there are 10,000 gallons of water in the pool.

   (a) How much water is in the tank at 5 pm? Your answer should involve a definite integral.

   (b) How much water is in the tank at 1 pm?

   (c) How much water is in the tank x hours after noon?

   (d) In this scenario, what does ${}_2A_f(x)$ represent?

   (e) Let $W(x)$ be the amount of water in the pool x hours after noon. How would you express "the instantaneous rate at which water is flowing into/out of the tank at 6 pm" in terms of W? In terms of f? In terms of ${}_2A_f$?

4. Let $f(x) = \dfrac{1}{1+x^2}$.

   (a) Sketch $f(x)$.

   Answer the following questions about ${}_0A_f(x)$; feel free to use Part 1 of FTOC as appropriate.

   (b) What is ${}_0A_f(0)$?

   (c) On what intervals is ${}_0A_f$ increasing? Decreasing? How do you know?

   (d) On what intervals is ${}_0A_f$ concave up? Concave down?

(e) Sketch ${}_0A_f$.

(f) Does your sketch remind you of a function you know? Could ${}_0A_f(x)$ actually be that function — would that be consistent with what FTOC 1 tells you about ${}_0A_f(x)$?

(g) Sketch ${}_1A_f$ on your graph in (e). How does ${}_1A_f$ relate to ${}_0A_f$?

56 Fundamental Theorem of Calculus Part 2

1. (Blue Bell revisited) At the Blue Bell ice cream factory, milk is stored in a 40,000 gallon tank. The rate of milk flowing into and out of the tank t hours after noon one day is $f(t) = 1000 - 300t^2$ gallons per hour. At 2 pm, there are 15,000 gallons of milk in the tank.

   (a) How much milk is in the tank at 5 pm? Express your answer in terms of a definite integral or area function.

   (b) Express the net change in the amount of milk between 1 pm and 3 pm as a definite integral.

   (c) Let $M(t)$ be the amount of milk in the tank at time t. What is $M'(t)$? What is $M(2)$? Using these two pieces of information, can you find a formula for $M(t)$?

   (d) Use your answer to (c) to find the net change in the amount of milk in the tank between 1 pm and 3 pm.

What you just found out was extremely important!

The Fundamental Theorem of Calculus, Part 2 (FTOC 2). If f is a continuous function on $[a, b]$, then $\int_a^b f(t)\,dt =$ ____________________________________ .

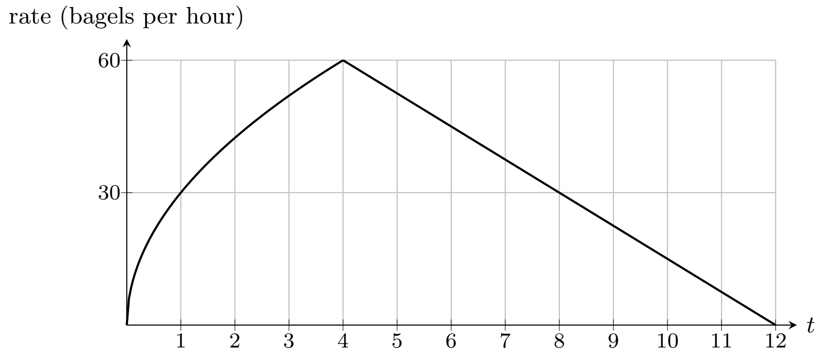
2. Evaluate $\int_{-2}^{2} 4x^3\,dx$.

3. Evaluate $\int_{4}^{3} x\,dx$.

4. A busy bagel shop starts making bagels at 6 am each morning. One day, the rate at which bagels are made is

$$r(t) = \begin{cases} 30\sqrt{t} & 0 \le t \le 4 \\ 7.5(12-t) & 4 < t \le 12 \end{cases} \text{ bagels per hour,}$$

where t is measured in hours after 6 am. (The graph of $r(t)$ is below.)



The bagel shop is open 7 am - 6 pm; on this day, customers come at a steady rate of 30 customers per hour between 7 am and 6 pm. Each customer orders one bagel and receives it immediately.

(a) At what times on this day is the number of bagels the shop has available increasing? At what times is it decreasing?

(b) At the end of the day, any unsold bagels are donated to a local food pantry. How many bagels are donated at the end of this day?

(c) What is the largest number of bagels the shop has available at any point during this day?

5. (a) Find two different antiderivatives of $f(t) = 3t^2 + 5$.

   (b) For each antiderivative you found in (a), use that antiderivative along with FTOC 2 to evaluate $\displaystyle\int_{-1}^{10} (3t^2 + 5)\, dt$.

   (c) In (b), you should have gotten the same answer using either antiderivative. Explain, in your own words, why this works.

6. Find the signed area under the curve $y = \sin x$ from $x = \frac{\pi}{2}$ to $x = \pi$. (Should your answer be positive or negative?)

57 Antiderivatives

1. In January 2010 (which we'll call $t = 0$), the price of a gallon of whole milk was \$3.24. Suppose that, after that, the price changed at a rate of $r(t) = 1 + 1.5 \sin t$ cents per month, where t is measured in months. How much did a gallon of whole milk cost 12 months later, in January 2011?

2. List all of the "basic" derivatives that you can.

We use the notation $\int f(x)\,dx$ to mean "all antiderivatives of f" or "the most general antiderivative of $f(x)$" and call this the <u>indefinite integral of $f(x)$</u>.

While the definite integral $\int_a^b f(x)\,dx$ is a number, the indefinite integral $\int f(x)\,dx$ is a family of functions.

3. **Basic Indefinite Integrals.** (*Know these!*)

   - When n is a constant other than -1, $\int x^n\,dx =$
   - $\int \frac{1}{x}\,dx =$
   - $\int e^x\,dx =$
   - When b is a positive constant, $\int b^x\,dx =$
   - $\int \sin x\,dx =$
   - $\int \cos x\,dx =$
   - $\int \sec^2 x\,dx =$
   - $\int \sec x \tan x\,dx =$
   - $\int \frac{1}{1+x^2}\,dx =$
   - $\int \frac{1}{\sqrt{1-x^2}}\,dx =$

4. Evaluate the following definite and indefinite integrals.

   (a) $\displaystyle\int \left(3e^u - u^2 + 5\right) du$

   (b) $\displaystyle\int \frac{5}{7x}\, dx$

   (c) $\displaystyle\int_{-e}^{-1/e} \frac{5}{7x}\, dx$

   (d) $\displaystyle\int (e^\pi + ey + \pi^y)\, dy$

   (e) $\displaystyle\int (\pi + x)\sqrt{x}\, dx$

   (f) $\displaystyle\int \left(\frac{e}{x^2} + 2^x \cdot 3^x\right) dx$

   (g) $\displaystyle\int_{-3}^{0} \left(\sqrt{9 - x^2} + x^2\right) dx$

5. On a camping trip, Ali sets out from his campsite at noon to wander along a trail. We'll describe positions along the trail by how many meters north of the campsite they are; for example, position -10 is 10 meters south of the campsite.

   Ali's velocity t minutes after noon is $v(t) = 3t^2 - 6t - 9$ meters per minute.

   (a) What is Ali's average velocity between noon and 12:04 pm? Remember that average velocity is defined to be $\dfrac{\text{change in position}}{\text{change in time}}$.

   (b) What is Ali's average *speed* between noon and 12:04 pm? Remember that average speed is defined to be $\dfrac{\text{distance traveled}}{\text{change in time}}$.

   (c) True or false: The average velocity of a moving object between time a and time b is simply the average of its velocities at times a and b.

6. True or false:

   (a) If F is an antiderivative of f and G is an antiderivative of g, then FG is an antiderivative of fg.

   (b) If F is an antiderivative of f and G is an antiderivative of g, then $F + G$ is an antiderivative of $f + g$.

58 Integration by Substitution

Some basic antiderivatives. Here are some of the basic antiderivatives you should know.

If $n \neq -1$, $\int u^n\,du =$ ________________ $\quad \int \cos u\,du =$ ________________

$\int \frac{1}{u}\,du =$ ________________ $\quad \int e^u\,du =$ ________________

$\int \sin u\,du =$ ________________ $\quad \int \frac{1}{1+u^2}\,du =$ ________________

1. **Warm-up.** Can you evaluate the following?

   (a) $\int_1^4 \frac{1}{3}\sqrt{u}\,du$ (b) $\int \cos(3x)\,dx$ (c) $\int \cos(x^2)\,dx$

2. Fill in the blank in each statement:

   (a) $\int$ ________________ $du = \sin(u) + C$

   (b) $\int$ ________________ $dx = \sin(x^2) + C$

3. Evaluate the following integrals using substitution.

   (a) $\int e^{\sin x} \cos x\,dx$

   (b) $\int \sin x \cos^5 x\,dx$

(c) $\displaystyle\int e^{-2x}\cos(e^{-2x})\,dx$

(d) $\displaystyle\int_1^{e^\pi} \frac{\sin(\ln x)}{5x}\,dx$

(e) $\displaystyle\int_1^4 x^2\sqrt{1+x^3}\,dx$

4. *Examples where substitution is useful even though there isn't an obvious composition.*

(a) $\displaystyle\int \tan x\,dx$

(b) $\displaystyle\int_{1/e}^{e^3} \frac{4\ln x}{x}\,dx$

5. One of the following integrals can easily be evaluated using substitution. Which one? Evaluate it.

   (a) $\displaystyle\int e^{-x^2}\,dx$ (b) $\displaystyle\int xe^{-x^2}\,dx$

6. Suppose g is a mysterious function, and the only thing you know about it is that $\displaystyle\int_0^{12} g(x)\,dx = \pi$. For each of the following integrals, decide whether you have enough information to evaluate the integral; if so, evaluate it.

   (a) $\displaystyle\int_0^{12} g(3x)\,dx$

   (b) $\displaystyle\int_0^{4} g(3x)\,dx$

   (c) $\displaystyle\int_0^{144} g(x^2)\,dx$

   (d) $\displaystyle\int_0^{2\sqrt{3}} xg(x^2)\,dx$

   (e) $\displaystyle\int_0^{144} xg(x^2)\,dx$

7. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{e^x}{\sqrt{2e^x+5}}\,dx$ (b) $\displaystyle\int \frac{x+1}{\sqrt[3]{3x^2+6x+5}}\,dx$

59 Integration by Substitution Continued

1. **Warm-Up.** Evaluate the following integrals; you should *not* need to use substitution, only algebra.

   (a) $\int (u-5)\sqrt{u}\,du$ (b) $\int \frac{(t+1)^2}{t}\,dt$ (c) $\int \frac{u+3}{\sqrt{u}}\,du$

2. *So far, when we've used substitution, we've generally tried to pick u so that we also see du in the integral. However, sometimes substitution is used simply to rewrite an integral in a friendlier form. This is the case in each of the following problems.*

   (a) $\int x\sqrt{x+5}\,dx$. (Hint: Let $u = x + 5$. What is x in terms of u?)

   (b) $\int \frac{2x}{\sqrt{x-3}}\,dx$

   (c) $\int \frac{x}{x+3}\,dx$

3. Evaluate each of the following integrals.

   (a) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

   (b) $\displaystyle\int_0^1 \frac{2x+3}{x^2+1}\,dx$

   (c) $\displaystyle\int y\cdot\sqrt[3]{2y+5}\,dy$

(d) $\displaystyle\int e^{3x}\sqrt{e^x+2}\,dx$

(e) $\displaystyle\int \frac{e^x}{e^{2x}+2e^x+1}\,dx$

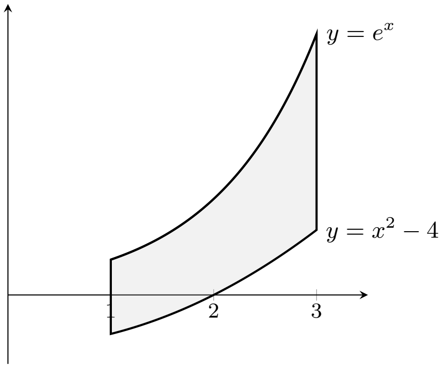
4. It is 10 am, and five ants have found their way into a picnic basket. Ants are notorious followers, so ants from all over the vicinity follow their five brethren into the basket. The culinary treat awaiting them is unsurpassed elsewhere, so once the ants find their way into the basket they choose not to leave. Suppose the rate at which ants are climbing into the basket is well modeled by $100e^{-0.2t}$ ants per hour, where $t=0$ is the benchmark hour of 10 am.

   (a) How many ants are in the basket at 1 pm?

   (b) How many ants are in the basket x hours after 10 am?

60 Slicing to find Area

1. Approximate the area under $y = e^x$, above $y = x^2 - 4$, to the right of $x = 1$, and to the left of $x = 3$.



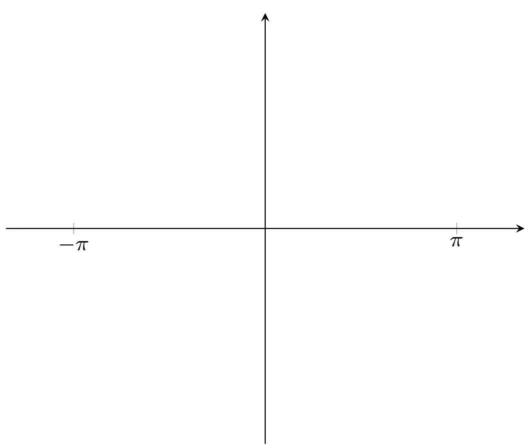
2. Find the area of the region bounded by $y = 2 - x^2$ and $y = x$.

3. Find the area enclosed by $y = -2x$, $y = x^3$, and $y = 1$.

4. Find the area under the curve $y = \ln x$ from $x = 1$ to $x = e$. (Please give your answer as a number, not an integral.)

5. Let $\mathcal{R}$ be the region enclosed by $y = \arccos x$, the x-axis, and $x = -\frac{1}{2}$. Find the area of $\mathcal{R}$. (Please evaluate your integral.)

6. (a) On the following set of axes, sketch the curves $y = \sqrt{3}\sin x$ and $y = \cos x$.



(b) In your picture, there should be a region which is enclosed by the two curves. Find the area of this region. (Please give a numerical answer, not an integral.)

61 Slicing to Solve Density Problems

1. Oil spill in Gulf of Mexico – day 1 A satellite image taken the day of an oil spill in the Gulf of Mexico revealed that the density of oil on the water was approximately 0.5 tons per square kilometer. From the satellite image, the area covered by the oil was roughly a square of side length 2 kilometers. What is approximately the total quantity of oil that was spilled? Note: Oil is less dense than water, so it stays at the surface of the water.

2. Oil spill in Gulf of Mexico – day 6

   On day 6 of the oil spill, the winds in the Gulf (blowing North) have spread the oil over a larger surface which is now roughly rectangular, with the northern and southern edges still of length 2 km and eastern and western edges of length 5 km. But a new satellite image reveals that the density of oil on the water is not constant anymore. In fact, this density is given by $\rho(x) = 0.5 - 0.02x^2$ tons per square kilometer. Here x is the distance from the southern edge of the rectangular region.

   (a) How would you approximate the total quantity of oil that was spilled?

   (b) What is the **exact** amount of oil spilled?

3. Petsi Pies is known for its chocolate pastries which are shaped like right triangles with sides of 6, 8, and 10 centimeters. Chocolate shavings are sprinkled on each piece so that the density of chocolate is given by $\rho(x)$ ounces per square centimeter, where x measures distance (in centimeters) from the 6 cm side of the pastry.

   (a) Show in a sketch the meaning of x. Then, show how you will **slice** the pastry to solve the problem.

   (b) How can you **approximate** the amount of chocolate in the k-th slice?

   (c) Write a Riemann **sum** that approximates the amount of chocolate on an entire pastry.

   (d) What is the exact amount of chocolate on each pastry?

4. An ecologist marks off a square piece of forest to research with side lengths 1000 m. A river runs along one diagonal of the forest. Assume that the number of trees per unit area depends only on the distance s from the river and is given by $\rho(s) = 1000 - s$ trees/m^2.

   (a) Show in a sketch the meaning of s. How will you **slice** the problem?

   (b) Write an integral that gives the total number of trees.

62 More Density Problems

1. Farmer Joe and Farmer Bob have identical circular corn fields of radius 100 meters. However, they have different ideas about how to water their fields.

   (a) Farmer Joe decides to install a single high-powered sprinkler right in the center of his corn field. Corn grows better near the sprinkler, so the yield density of corn in Farmer Joe's field is given by $j(x) = \dfrac{10000}{1+x^2}$ stalks of corn per square meter, where x is distance from the sprinkler. We'd like to find out how much corn Farmer Joe's field yields.

      i. Show in a sketch the meaning of x in Farmer Joe's field. Then show in a sketch how you would **slice** Farmer Joe's field.

      ii. How would you **approximate** the corn yield of the k-th slice?

      iii. Write a general Riemann **sum** that approximates the total yield of Farmer Joe's corn field.

      iv. By **taking an appropriate limit**, find a definite integral that gives the exact yield of Farmer Joe's field.

(b) Farmer Bob decides to install a 200 meter long straight irrigation pipe in his corn field. The pipe runs right down the center of the plot, and the corn density in Farmer Bob's plot varies with distance to the pipe. If the density is given by $b(x) = 200 - x$ corn stalks per square meter, where x is distance to the pipe, write a definite integral that gives the number of corn stalks in Farmer Bob's field.

2. A cone of Hawaiian shave[14] ice is deliciously drizzled with fresh passion fruit purée. The cone has a height of 8 cm and radius of 6 cm is filled flush with the top.

   The density of the purée varies with the vertical distance to the tip of the cone, as some time has passed and a lot of the purée has settled at the bottom. Let $\rho(x) = \frac{1}{x+1}$ be the number of ounces of purée per cubic centimeter, where x is the vertical distance from the tip.

   (a) Slice up the cone into n equal slices so that each slice has about the same density of purée. How much purée is in the k-th slice?

   (b) Write an integral giving the total amount of purée in the cone.

3. *Variations.*

   (a) Suppose that the density of purée in #2 had been given by $q(v) = \frac{1}{v+1}$ ounces of purée per cubic centimeter, where v is vertical distance from the top of the cone. In this case, the purée has been recently applied, and there is more at the top than at the bottom. How much purée would there be in the cone? (You may leave your answer as a definite integral.)

   (b) Suppose that the cone in (a) had been filled only to a depth of 7 cm. Then, how much purée would there be in the cone?

[14] The iced is shaved, but locals call it "shave ice."

63 Introduction to Differential Equations

A differential equation is an equation that involves an unknown function and one or more of its derivatives. Differential Equations are a powerful tool for modeling. It is often simpler to describe a phenomenon from the perspective of rates of change.

1. Let $B(t)$ model the number of bacteria on a petri dish t minutes after you start an experiment.

   (a) What factors do you think contribute to the growth rate, $B'(t)$?

   (b) Write down a differential equation that encapsulates your answers to (a).

2. Let's look at the equation $x = x^2 + 3x + 2$.

   (a) A solution to this equation is a _______.

   (b) Which of the following are solutions to the equation?

   A. $x = -2$ B. $x = 3$ C. $x = -1$

3. Let's look at the differential equation $\dfrac{dq}{dt} = -q^2$.

   (a) A solution to the differential equation is a _______. Write the differential equation in words.

   (b) Which of the following functions are solutions of the differential equation?

   A. $q(t) = -\frac{t^3}{3}$ C. $q(t) = \frac{1}{t+2}$ E. $q(t) = \sqrt{t}$

   B. $q(t) = \frac{1}{t}$ D. $q(t) = \frac{1}{2t+1}$ F. $q(t) = 0$

   (c) Can you guess any more solutions of the differential equation?

4. (a) When we see the differential equation $y' = 3t^2 - 4t$, we know we are looking for ________.

   (b) Which of the following are solutions of the differential equation $y' = 3t^2 - 4t$?

   A. $y = t^3$ B. $y = t^3 - 2t^2 + 3$ C. $y = 0$

   (c) Can you guess any more solutions of $y' = 3t^2 - 4t$? Can you find them all? (In other words, can you find the <u>general solution</u> of $y' = 3t^2 - 4t$?)

5. (a) Can you think of any solutions of the differential equation $\dfrac{dP}{dt} = 2t$? Can you find all solutions? (In other words, can you find the general solution?)

   (b) Can you think of any solutions of the differential equation $\dfrac{dP}{dt} = 2P$? Can you find the general solution?

Very important fact. If k is a constant, then the general solution of $\dfrac{dy}{dt} = ky$ is: $y(t) = Ce^{kt}$.

6. (Problem 1, revisted) The population of a bacteria colony increases at a rate proportional to the population size. At $t = 0$, the colony has a population of 500 bacteria and the colony is growing at a rate of 100 bacteria per hour.

   (a) Write a differential equation that models the bacteria population.

   (b) Find a solution to your differential equation from part (a).

7. Which of the following are solutions to the differential equation $\dfrac{d^2y}{dx^2} = 2x\left(\dfrac{dy}{dx}\right)^2$?

   A. $y = \frac{1}{x}$ B. $y = 17$ C. $y = -\arctan(x)$ D. $y = \frac{1}{x} + 3$

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1. Problem Set 32, #6).

   Walter puts some money in a bank account that earns a nominal annual interest rate of 3%, compounded continuously. Let $M(t)$ be the amount of money he has after t years.

   (a) Write a differential equation about $M(t)$ that expresses the fact that the money is earning 3% annual interest, compounded continuously.

   (b) Interpret your differential equation in words.

Key Idea 1. Saying that two quantities A and B are proportional to each other means that one is a constant multiple of the other. That is, $A = kB$ for some constant k.

2. One way that psychologists study how people learn is by asking them to learn a list of nonsense words.[15] Suppose that $L(t)$ is the percent of such a list that a particular person has learned at time t, where t is measured in days.

   (a) A simple model for learning is to assume that learning happens at a rate proportional to the percent of the list left to be learned. Write a differential equation that expresses this. Can you write an initial condition?

   (b) A student who has learned a fifth of the list already is currently learning at a rate of 10% of the list per day. Incorporate this into your model.

[15] This is meant to be a "pure" experiment, in that the learners won't have any prior knowledge that would help them with the task.

3. A juicy rumor is spreading through Harvard's freshman class, which has 1600 students. The rumor spreads at a rate proportional to the product of the number of freshmen who have already heard it and the number of freshmen who have not yet heard it. At time $t = 0$, 5 freshmen have heard the rumor.

   (a) If $R(t)$ is the number of people who have heard the rumor at time t, write a differential equation that models the situation. Can you write an initial condition?

   (b) Lucy thinks that the rumor should instead spread at a rate proportional to the number of students who haven't heard the rumor yet. Which model (Lucy's or the original one) do you think is more reasonable? Why?

> **Key Idea 2.** Differential equations can be used to model more complicated situations. When we want to do this, a key equation is
>
> rate of change = (rate in) − (rate out)
>
> *or* rate of change = (rate of increase) − (rate of decrease)

4. A drug is being administered to a patient at a constant rate of 10 mg/hr. The patient metabolizes and eliminates the drug at a rate proportional to the amount in his body. Let $M = M(t)$ be the amount (in mg) of medicine in the patient's body at time t, where t is measured in hours. Write a differential equation that models the situation.

5. Dante borrows \$150,000 from the bank. He pays money back at a rate of \$1000 per month, while the bank charges him interest at a nominal annual interest rate of 6% per year compounded continuously.

   (a) Write a differential equation that models $B(t)$, the balance Dante owes the bank at time t, where t is measured in years and $t = 0$ is the time at which Dante borrows the money. (*Be careful with units!*)

   (b) Which of the following gives Dante's balance at time t?

      i. $B(t) = 150{,}000e^{0.06t} - 12{,}000t$

      ii. $B(t) = 200{,}000 - 50{,}000e^{0.06t}$

      iii. $B(t) = (150{,}000 - 12{,}000t)e^{0.06t}$

6. A baker has 1800 mm^3 of bread dough which he will allow to rise. Under normal circumstances, the dough expands at a rate proportional to its current volume; after 1 hour of rising, it would multiply to be e times its original size. However, on this day, mice discover the bread dough immediately after the baker sets it out to rise; they nibble on it at a constant rate of 20 mm^3 per minute. Write a differential equation for $B(t)$, the volume of the bread dough t minutes after the baker sets it out to rise. Can you also write an initial condition?

7. Suppose we want to model the velocity of a falling snowflake. The physical law that governs the motion of objects is Newton's second law $F = ma$, which says that the total force on an object is given by the mass of the object multiplied by its acceleration.

   A snowflake is subject to two main forces: the force due to gravity (which is mass times the constant acceleration of 9.8 m/s^2) and air resistance, which can be reasonably modeled as being proportional to velocity. (This is a good model only for less dense objects.)

   Use this model to write a differential equation for $v(t)$, the velocity in m/s of a falling snowflake at time t, where t is measured in seconds.