Math 55a Optional Warm-Up

- This is **not** an actual homework assignment. It is a collection of warm-up problems, some serving as a pretext to review set theory, others more challenging. Please do not turn in; but keep your solutions handy, as some of these may appear on the first assignment.
- Recommended reading for set theory review: Halmos, *Naive set theory*, chapters 4–10.
- Questions marked * are harder. It's Math 55 after all. But don't worry if you have no idea how to get started on those.
- You are encouraged to discuss the homework problems with other students. Start using the `#homework` channel on Slack to ask questions, or `#studygroups` to connect with others and work on a problem together.
- If you have way too much time on your hands, start learning how use LaTeX to typeset beautiful math documents! Sign up for Overleaf at `https://www.overleaf.com/edu/harvard` and see the tutorials at `https://www.overleaf.com/learn`

Material covered: Set theory review (Halmos, chapters 4-10).

1. Let $f : X \to Y$ be a map of sets. A function $g : Y \to X$ is called a left (resp. right) inverse of f if $g \circ f = \mathrm{id}_X$ (resp. $f \circ g = \mathrm{id}_Y$).

(a) Show that f admits a right inverse if and only if f is surjective (*onto* in Halmos' terminology). Show that, in general, a right inverse is not unique.

(b) Show that f admits a left inverse if and only if f is injective (*one-to-one* in Halmos' terminology). Show that, in general, a left inverse is not unique.

2. Let $f : X \to Y$ be a map of sets.

(a) For $B \subset Y$ we define $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$. Show that

$$f^{-1}(Y-B) = X - f^{-1}(B), \quad f^{-1}\Big(\bigcup_{i\in I} B_i\Big) = \bigcup_{i\in I} f^{-1}(B_i), \quad \text{and } f^{-1}\Big(\bigcap_{i\in I} B_i\Big) = \bigcap_{i\in I} f^{-1}(B_i).$$

(b) For $A \subset X$ we define $f(A)$ to be the subset of Y consisting of all elements $y \in Y$ for which there exists $x \in A$ with $f(x) = y$. How does $f(X - A)$ compare with $Y - f(A)$? How does $f\Big(\bigcup_{i\in I} A_i\Big)$ compare with $\bigcup_{i\in I} f(A_i)$? How does $f\Big(\bigcap_{i\in I} A_i\Big)$ compare with $\bigcap_{i\in I} f(A_i)$?

3. Give an explicit bijection between $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$.

4. At his death, a millionaire left his 10 children a million dollars in cash, all in \$100, \$10, \$1 bills, 10-cent, and 1-cent coins. Show that there is a way for them to split the fortune into ten stacks of equal value. (Note that this would not be true if there were \$3 bills).

5*. Given a set S, let $\mathcal{E} \subset \mathcal{P}(S)$ be such that (1) $S \in \mathcal{E}$, (2) if $A \in \mathcal{E}$ then $S - A \in \mathcal{E}$, (3) if $A, B \in \mathcal{E}$ then $A \cup B \in \mathcal{E}$ and $A \cap B \in \mathcal{E}$.

Prove that if S is finite then there is a set T and a surjective map $f : S \to T$ such that $\mathcal{E} = \{f^{-1}(A),\ A \subset T\}$. What happens if S is infinite?