Last time, we studied operators $\varphi: V \to V$ (finite dim. vect space), their eigenspaces $Ker(\varphi-\lambda I)$ and generalized eigenspaces $V_\lambda = Ker(\varphi-\lambda I)^n$ ($n = \dim V$). We showed:

1) V_λ are invariant subspaces of φ and linearly independent inside V.
2) If k is alg. closed then $V = \bigoplus_{\lambda \text{ eigenvalue}} V_\lambda$.
3) For a nilpotent operator ($\varphi^m = 0$ for some $m>0$), $\exists$ Jordan basis $\{\varphi^{m_1}(v_1), \varphi^{m_1-1}(v_1), \dots, v_1, \dots, \varphi^{m_k}(v_k), \dots, v_k\}$ where $\varphi^{m_i+1}(v_i) = 0 \ \forall i$ in which the matrix of φ is block diagonal built from nilpotent Jordan blocks $\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$

We now combine our results to arrive at the

Jordan normal form: V finite dim. vector space over k alg. closed (e.g. $\mathbb{C}$), $\varphi \in Hom(V,V)$
$\Rightarrow \exists$ basis of V in which the matrix of φ is block-diagonal, with each block a Jordan block $\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$.

Rmks: • size 1 Jordan block: (λ), size 2: $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, ... φ is diagonalizable $\Leftrightarrow$ all the blocks have size 1.
- the values of λ that appear are exactly the eigenvalues of φ. There may be several blocks with the same λ; their direct sum is the generalized eigenspace V_λ.
- proof: we've seen $V = \bigoplus V_\lambda$ generalized eigenspaces; now $\varphi_{|V_\lambda} - \lambda I$ is nilpotent, so can decomposed into nilpotent Jordan blocks $\varphi_{|V_\lambda} - \lambda I = \bigoplus \begin{pmatrix} 0 & 1 & \\ & \ddots & 1 \\ & & 0 \end{pmatrix}$, so $\varphi_{|V_\lambda} = \bigoplus \begin{pmatrix} \lambda & 1 & \\ & \ddots & 1 \\ & & \lambda \end{pmatrix}$.

---

* Characteristic polynomial, minimal polynomial:

Let k be algebraically closed, $\varphi: V \to V$, $V = \bigoplus_{i=1}^{\ell} V_{\lambda_i}$ V_{λ_i} generalized eigenspaces

Call • $n_i = \dim V_{\lambda_i}$ the multiplicity of λ_i ($\sum n_i = \dim V$)
• m_i = nilpotence order of $(\varphi_{|V_{\lambda_i}} - \lambda_i Id)$ ie. smallest m_i st. $V_{\lambda_i} = Ker(\varphi - \lambda_i I)^{m_i}$

From the above: $m_i \le n_i$, and V_{λ_i} is diagonalizable iff all $m_i = 1$.

Def: • The characteristic polynomial of φ is $\chi_\varphi(x) = \prod_{i=1}^{\ell} (x-\lambda_i)^{n_i}$

The usual definition, once we have defined determinant, is: $\chi_\varphi(x) = \det(xI - \varphi)$

Manifestly, in a basis where $M(\varphi)$ is triangular (or Jordan), $M(xI-\varphi) = \begin{pmatrix} x-\lambda_1 & & * \\ & \ddots & \\ & & x-\lambda_n \end{pmatrix}$ and this is the same thing. (but can use any basis to calculate det).

The significance is: given matrix of φ in any basis, A, we can calculate $\chi(x) = \det(xI - A) \in k[x]$, and solve for roots = eigenvalues

multiplicities = dim. of gen^d eigenspaces

(This also works over non alg. closed k, without any guarantee that $\chi(x)$ has any roots.)

Def: The <u>minimal polynomial</u> of φ is $\mu_\varphi(x) = \prod_{i=1}^{\ell} (x-\lambda_i)^{m_i}$.

<u>Significance</u>: $(\varphi - \lambda_i)^k = 0$ on the gen. eigenspace V_{λ_i} iff $k \geq m_i$ & invertible on the other gen^d eigenspaces.

So $\mu_\varphi(\varphi)$ = simplest polynomial expression in φ that is zero on all V_{λ_i}'s, hence on $\bigoplus V_{\lambda_i} = V$.

Hence: $\mu_\varphi(\varphi) = 0$, and $\forall p \in k[x]$, $p(\varphi) = 0 \in \mathrm{Hom}(V,V)$ iff μ_φ divides p.

Since nilpotence order m_i is always $\leq \dim V_{\lambda_i} = n_i$, μ_φ divides χ_φ, so:

<u>Thm</u> (Cayley-Hamilton) $\chi_\varphi(\varphi) = 0$.

(This is also true over non alg. closed k, by passing to alg. closure; see below for an example)

- A word about operators on finite dim. <u>$\mathbb{R}$-vector spaces</u>:

Let V real vector space (dim. n), $\varphi : V \to V$ linear operator.

Since $\mathbb{R}$ is not alg. closed, φ might not have eigenvalues, and we can't put φ in triangular or Jordan form.

<u>Yet</u>: every real operator has an invariant subspace of dim. <u>1</u> or 2

<u>Approach</u>: work over $\mathbb{C}$ which is alg. closed. How do we do this?

Def: The <u>complexification</u> of V is $V_\mathbb{C} = V \times V = \{v + iw \mid v, w \in V\}$,

with addition $(v_1 + iw_1) + (v_2 + iw_2) = (v_1 + v_2) + i(w_1 + w_2)$

scalar mult. $(a+ib)(v+iw) = (av - bw) + i(bv + aw)$ $a, b \in \mathbb{R}$

- This is a $\mathbb{C}$-vector space of dimension n: if $(e_1, \dots e_n)$ is a basis of V over $\mathbb{R}$, then $e_1 (= e_1 + i0), \dots, e_n$ is also a basis of $V_\mathbb{C}$ over $\mathbb{C}$.

- Given $\varphi : V \to V$ $\mathbb{R}$-linear, we can extend it to $\varphi_\mathbb{C} : V_\mathbb{C} \to V_\mathbb{C}$ $\mathbb{C}$-linear simply by $\varphi_\mathbb{C}(v + iw) = \varphi(v) + i\varphi(w)$. Choosing a basis $(e_1, \dots e_n)$ as above, the matrix of $\varphi_\mathbb{C}$ is the <u>same</u> as that of φ $(\varphi_\mathbb{C}(e_j + i0) = \varphi(e_j) + i0)$.

But now... $\varphi_{\mathbb{C}}$ is guaranteed to have an eigenvector!
(and gen^d eigenspaces, and Jordan form, ...)

Let $\mathbb{v} = v + iw$ be an eigenvector of $\varphi_{\mathbb{C}}$ for eigenvalue $\lambda \in \mathbb{C}$, $\varphi_{\mathbb{C}}(\mathbb{v}) = \lambda \mathbb{v}$.

There are two cases:

- if $\lambda \in \mathbb{R}$, then $\varphi_{\mathbb{C}}(v+iw) = \varphi(v) + i\varphi(w) = \lambda v + i\lambda w$
  $\Rightarrow v = \mathrm{Re}(\mathbb{v})$ and $w = \mathrm{Im}(\mathbb{v})$ are eigenvectors of φ with the same eigenvalue λ (if they are nonzero; one of them is).
  (⚠ the multiplicity of λ for φ has no reason to be even).
- if $\lambda = a + ib \notin \mathbb{R}$, then $\varphi_{\mathbb{C}}(v+iw) = (a+ib)(v+iw)$
  $\Rightarrow \varphi_{\mathbb{C}}(v-iw) = (a-ib)(v-iw)$ (compare real and imaginary parts!)
  i.e. $\overline{\mathbb{v}} = v - iw$ is an eigenvector of $\varphi_{\mathbb{C}}$ with eigenvalue $\overline{\lambda}$.
  It follows that v and w are linearly independent, and span a 2-dimensional invariant subspace $U \subset V$: $\quad \begin{aligned}\varphi(v) &= av - bw \\ \varphi(w) &= bv + aw\end{aligned} \quad \mathcal{M}(\varphi_{|U}, (v,w)) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

(One could further study block triangular decompositions of φ etc. starting from $\varphi_{\mathbb{C}}$).

<u>Interlude</u>: the language of categories. (then we'll return to (bi)linear algebra)

<u>Def</u>: A <u>category</u> is a collection of <u>objects</u> + for each pair of objects, a collection of <u>morphisms</u> $\mathrm{Mor}(A,B)$, and an operation called <u>composition</u> of morphisms,
$$\begin{aligned}\mathrm{Mor}(A,B) \times \mathrm{Mor}(B,C) &\longrightarrow \mathrm{Mor}(A,C) \quad \text{st.}\\ f\ ,\ g &\longmapsto g\circ f\end{aligned}$$
1) every object A has an <u>identity</u> morphism $id_A \in \mathrm{Mor}(A,A)$ st. $\forall f \in \mathrm{Mor}(A,B)$, $f \circ id_A = id_B \circ f = f$.
2) composition is <u>associative</u>: $(f\circ g)\circ h = f \circ (g \circ h)$.

<u>Ex</u>: 1) category of sets, $\mathrm{Mor}(A,B)$ = all maps $A \to B$
2) Vect_K finite-dim^l vector spaces / K, linear maps.
3) groups, group homomorphisms
4) top. spaces, continuous maps.

<u>Def</u>: $f \in \mathrm{Mor}(A,B)$ is an <u>isomorphism</u> if $\exists g \in \mathrm{Mor}(B,A)$ st. $g \circ f = id_A$ and $f \circ g = id_B$.
(the inverse isomorphism)

<u>Check:</u> • the inverse of f, if it exists, is unique..
• id_A is an isomorphism; f iso $\Rightarrow f^{-1}$ iso; f, g isos $\Rightarrow g\circ f$ isos.

$\rightsquigarrow$ The <u>automorphisms</u> of A, $Aut(A) = \{\text{isomorphisms } A \to A\} \subset Mor(A,A)$, form a <u>group</u>.

• Isomorphic objects have isomorphic automorphism groups: an isomorphism $f \in Mor(A,B)$ determines an isom. of groups $c_f : Aut(A) \to Aut(B)$, $g \mapsto f\circ g \circ f^{-1}$.

<u>Ex:</u> 1) In Sets, A finite set with n elements $\Rightarrow Aut(A) = \{\text{bijections } A \to A\} \cong \mathfrak{S}_n$
2) V = n-dim! vector space/k $\Rightarrow Aut(V) \cong GL_n(k)$ invertible $n\times n$ matrices

---

★ <u>Products and sums in categories:</u>

• Given objects A, B in a category $\mathcal{C}$, a <u>product</u> $A\times B$ is an object Z of $\mathcal{C}$ and a pair of maps $\pi_1 : Z \to A$, $\pi_2 : Z \to B$ st. $\forall T \in ob\,\mathcal{C}$, $\forall f_1 \in Mor(T,A)$, $f_2 \in Mor(T,B)$, $\exists!$ (unique) $\varphi \in Mor(T,Z)$ st. $\pi_1 \circ \varphi = f_1$ and $\pi_2 \circ \varphi = f_2$.

$$\begin{tikzcd} & T \arrow[dl, bend right, "f_1"'] \arrow[d, dashed, "\exists!\varphi"] \arrow[dr, bend left, "f_2"] & \\ A & Z \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

<u>Ex.</u> in Sets, $Z = A\times B$ usual Cartesian product
π_1, π_2 projection maps
given $f_1 : T \to A$, $f_2 : T \to B$, def. $\varphi : \begin{matrix} T \to A\times B \\ t \mapsto (f_1(t), f_2(t)) \end{matrix}$

Same in Groups, $Vect_k$

• A <u>sum</u> of objects A and B is an object Z of $\mathcal{C}$ + maps $i_1 : A \to Z$, $i_2 : B \to Z$ st. $\forall T \in ob\,\mathcal{C}$, $\forall f_1 \in Mor(A,T)$, $\forall f_2 \in Mor(B,T)$, $\exists! \varphi \in Mor(Z,T)$ st. $\varphi \circ i_1 = f_1$ & $\varphi \circ i_2 = f_2$.

$$\begin{tikzcd} & T & \\ A \arrow[ur, bend left, "f_1"] \arrow[r, "i_1"'] & Z \arrow[u, dashed, "\varphi"] & B \arrow[l, "i_2"] \arrow[ul, bend right, "f_2"'] \end{tikzcd}$$

<u>Ex:</u> in Sets, this is $Z = A \sqcup B$ disjoint union; define $\varphi : Z \to T$
$x \mapsto \begin{cases} f_1(x) & \text{if } x\in A \\ f_2(x) & \text{if } x \in B. \end{cases}$

in $Vect_k$, it's $Z = A \oplus B$ (so... sum = product!)
with i_1, i_2 = inclusion of $\begin{matrix} A & \text{as} & A\oplus 0 \subset Z \\ B & & 0 \oplus B \subset Z \end{matrix}$ define $\varphi : Z \to T$
$(a,b) \mapsto f_1(a) + f_2(b)$.

etc...

★ <u>Functors:</u>

<u>Def:</u> C, D categories. A (covariant) <u>functor</u> $F : C \to D$ is an assignment
• to each object X in C, an object $F(X)$ in D.
• to each morphism $f \in Mor_C(X,Y)$, a morphism $F(f) \in Mor_D(F(X), F(Y))$
st. 1) $F(id_X) = id_{F(X)}$
2) $F(g\circ f) = F(g) \circ F(f)$.

Ex: 1) forgetful functor taking a group, a top. space, ... to the underlying set.

2) on vector spaces, given a vect. space V. $F: W \longmapsto \mathrm{Hom}(V,W)$

if $f: W \to W'$ is linear, then induced map $\mathrm{Hom}(V,W) \xrightarrow{F(f)} \mathrm{Hom}(V,W')$

This gives a functor $\mathrm{Vect}_k \to \mathrm{Vect}_k$ (denoted $\mathrm{Hom}(V,\cdot)$) $a \longmapsto f \circ a$.

3) Complexification, $\mathrm{Vect}_{\mathbb{R}} \to \mathrm{Vect}_{\mathbb{C}}$: on objects, $V \mapsto V_{\mathbb{C}}$, on morphisms $\varphi \mapsto \varphi_{\mathbb{C}}$ see above.

3) Sets $\to$ Groups

$X \mapsto$ free group generated by X. Eg. $F(\{a,b\}) = \langle a, b \rangle$ free group on 2 generators.

* A contravariant functor = same except direction of morphisms is reversed:

$f \in \mathrm{Mor}_C(X,Y) \mapsto F(f) \in \mathrm{Mor}_D(F(Y), F(X))$; $F(g \circ f) = F(f) \circ F(g)$.

Ex: on Vect_k, $V \mapsto V^*$ dual (see HW5).