Math 55a Lecture 14 - Monday 10/5/20 - Midterm feedback: see Canvas announcement

Recall: we're studying operators on an inner product space $(V, \langle\cdot,\cdot\rangle)$
V real vector space (finite-dim.),
$\langle\cdot,\cdot\rangle : V\times V \to \mathbb{R}$ symmetric definite positive bilinear form $\begin{cases} \langle u,v\rangle = \langle v,u\rangle & \forall u,v \\ \langle u,u\rangle \geq 0 & \forall u \\ \langle u,u\rangle = 0 \Leftrightarrow u=0. \end{cases}$

Def: Say $T: V\to V$ is an <u>orthogonal operator</u> if it respects the inner product, ie. $\langle Tu, Tv\rangle = \langle u,v\rangle \ \forall u,v\in V$.

(In other terms, T "preserves lengths and angles"; maps orthonormal bases ($\langle e_i, e_j\rangle = \delta_{ij}$) to orthonormal bases ($\langle Te_i, Te_j\rangle = \delta_{ij}$).

δ_{ij}: 1 if $i=j$, 0 else

Def: Say T is <u>self-adjoint</u> if $\langle Tu, v\rangle = \langle u, Tv\rangle \ \forall u,v\in V$.

Def: Let $T: V\to V$ linear operator on an inner product space. $(V, \langle\cdot,\cdot\rangle)$
There exists a unique linear operator $T^*: V\to V$, called the <u>adjoint</u> of φ, such that $\langle v, T(w)\rangle = \langle T^*(v), w\rangle \ \forall v,w\in V$.

<u>Indeed</u>: given $v\in V$, the linear functional
$$V \longrightarrow \mathbb{R}$$
$$w \longmapsto \langle v, T(w)\rangle$$
is, using nondegeneracy of $\langle\cdot,\cdot\rangle$, given by the inner product of w with some element of V, which we call $T^*(v)$; then check this has linear dependence on v.

<u>Equivalently</u>: $\langle\cdot,\cdot\rangle$ defines an isom. $\varphi: V \xrightarrow{\sim} V^*$. Then T^* is the composition of
$$V \xrightarrow{\ \varphi\ } V^* \xrightarrow{\ T^t\ } V^* \xrightarrow{\ \varphi^{-1}\ } V$$
$$v \longmapsto \langle v, \cdot\rangle \mapsto \langle v, T(\cdot)\rangle = \langle T^*(v), \cdot\rangle \mapsto T^*(v).$$

Def: $T: V\to V$ is <u>self-adjoint</u> if $T^* = T$. (ie. $\langle v, Tw\rangle = \langle Tv, w\rangle \ \forall v,w$).

* In an orthonormal basis $(e_1, \dots, e_n)$ of V, $\langle v,w\rangle = \underset{\text{transpose gives a row vector}}{V^t}\ \underset{\text{column vector}}{W}$, so if matrix of T is M, T^* is N,

$$\left.\begin{aligned} \langle v, T(w)\rangle &= v^t M w \\ \langle T^*(v), w\rangle &= (Nv)^t w = v^t N^t w \end{aligned}\right\} \Rightarrow \text{comparing: } N^t = M, \text{ so } N = M^t.$$

Hence: $\mathcal{M}(T^*) = \mathcal{M}(T)^t$ in orthonormal basis; T is self-adjoint $\Leftrightarrow \mathcal{M}(T)$ <u>symmetric</u>

Note that self-adjoint operators ($\sim$symmetric matrices) need not be invertible.
For example 0 is a self-adjoint operator ...

Prop: if T is self-adjoint and $S \subset V$ is an invariant subspace ($T(S) \subset S$) then $S^\perp$ is also an invariant subspace ($T(S^\perp) \subset S^\perp$)

Pf: Let $v \in S^\perp$, then $\forall w \in S$, $T(w) \in S$, so $\langle Tv, w\rangle \underset{\uparrow \atop (T^* = T)}{=} \langle v, Tw\rangle \underset{\uparrow \atop (v \in S^\perp,\, Tw \in S)}{=} 0$.

Since $\langle Tv, w\rangle = 0$ $\forall w \in S$, we get: $Tv \in S^\perp$. □

Lemma: If T is self-adjoint then $\forall a \in \mathbb{R}_+$, $T^2 + a$ is invertible.

Pf: $\forall v \in V - \{0\}$,

$$\begin{aligned}\langle (T^2+a)v, v\rangle &= \langle T^2 v, v\rangle + a\langle v, v\rangle \\ &= \langle Tv, Tv\rangle + a\langle v,v\rangle = \|Tv\|^2 + a\|v\|^2 > 0\end{aligned}$$

So $(T^2+a)v \neq 0$. Hence $\ker(T^2+a) = 0$. □

Corollary: If $p \in \mathbb{R}[x]$ is a quadratic without real roots and $T^* = T$ then $p(T)$ is invertible.

Pf: enough to show $T^2 + bT + c$ is invertible whenever $b^2 - 4c < 0$.

write $T^2 + bT + c = (T + \frac{b}{2})^2 + a$, $a = c - \frac{b^2}{4} > 0$, $T + \frac{b}{2}$ self-adjoint

$\Rightarrow$ by the lemma (applied to $T + \frac{b}{2}$) this is invertible. □

$\Rightarrow$ **Theorem** (the spectral theorem for real self adjoint operators)

If $T: V \to V$ is self adjoint then T is diagonalizable, with real eigenvalues.
Even more, T can be diagonalized in an orthonormal basis of $(V, \langle \cdot, \cdot \rangle)$!

Pf:
- First we show the existence of an eigenvector.

  Pick $v \in V$, $v \neq 0$; since $v, Tv, \dots, T^n v \in V$ are linearly dependent ($n = \dim V$), there exists a (nonconstant) polynomial st. $(a_n T^n + \dots + a_0) v = 0$.

  This doesn't factor into degree 1 factors over $\mathbb{R}$ like it would over $\mathbb{C}$, but it factors into linear and quadratic factors

$$\underset{\uparrow \atop \text{these are the real roots}}{\prod (T - \lambda_i)} \;\; \underset{\uparrow \atop \text{irreducible (no real roots) coming from pairs of complex conjugate roots.}}{\prod (T^2 + b_j T + c_j)} \, v = 0$$

  At least one of these operators must have a nontrivial kernel (else their product is invertible, but $v \mapsto 0$!). By the previous corollary, each $T^2 + b_j T + c_j$ is invertible, so in fact some $T - \lambda_i$ must have a nontrivial kernel, hence an eigenvector!

- Now, diagonalization: we know there's an eigenvector $v_1 \in V$ with eigenvalue $\lambda_1 \in \mathbb{R}$; scaling v_1 if needed we may assume $\|v_1\| = 1$.

Then $S = \mathrm{span}(v) \subset V$ is an invariant subspace, hence (by Prop above) so is $S^\perp$.
By induction, using inner product on $S^\perp$ induced by restricting $\langle .,. \rangle$ and observing $T_{|S^\perp}$ is still self-adjoint, there is a basis of $S^\perp$, $(v_2 \dots v_n)$ (orthonormal if we wish), st. each v_j is an eigenvector of T.
Then $(v_1 \dots v_n)$ is a basis of V in which T diagonalizes, and we can assume it is orthonormal. □

So: $\boxed{T \text{ selfadjoint} \rightsquigarrow \mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \text{ in a suitable orthonormal basis.}}$

Rmk: this also implies: eigenvectors of T for distinct eigenvalues are orthogonal! but we already knew this because

$$Tv = \lambda v,\ Tw = \mu w \Rightarrow \lambda \langle v,w\rangle = \langle Tv, w\rangle = \langle v, Tw\rangle = \mu \langle v, w\rangle,$$

so $\lambda \neq \mu \Rightarrow v \perp w$.

Back to orthogonal transformations

($T: V \to V$ orthogonal $\Leftrightarrow \langle Tu, Tv\rangle = \langle u, v\rangle\ \forall u, v$
$\Leftrightarrow T^* = T^{-1}$)

Do we have a similar structure result?

→ in dim 1: T is mult. by a scalar, so T orthogonal $\Leftrightarrow T = \pm I$.

→ in dim 2: T orthogonal $\Leftrightarrow$ T is a rotation or a reflection.

(given orthonormal basis (e_1, e_2), Te_1 is any unit vector $\in$ unit circle $\{v \in V / |v| = 1\} = \{\cos\theta\, e_1 + \sin\theta\, e_2\}$; Te_2 is also unit vector and $\perp Te_1 \Rightarrow$ 2 possibilities

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \text{rotation by } \theta.$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \text{reflection}$$

Rotations have no eigenvectors
Reflections have eigenvalues ± 1
two orthogonal eigenspaces

Notation for $(V, \langle .,. \rangle)$, $SO(V) \subset \underset{\text{orthogonal}}{O(V)} \subset \underset{\text{invertible linear operators } T: V \to V}{GL(V)}$ subgroups

$SO(V)$: subgroup of orientation-preserving orthogonal transformations: those with $\det = +1$.
in dim 1: $\{+I\}$, in dim 2: rotations

Since $\underset{\langle . \rangle}{V} \simeq \underset{\text{std}}{\mathbb{R}^n}$ by choosing orthonormal basis, usually write $O(\mathbb{R}^n) = O(n)$
$SO(\mathbb{R}^n) = SO(n)$

$SO(n)$ has index 2 in $O(n)$, $\quad 1 \to SO(n) \to O(n) \xrightarrow[\det]{} \{\pm 1\} \simeq \mathbb{Z}/2 \to 1$.

$SO(2) \simeq S^1$ (rotations $\leftrightarrow$ angles)

Recall : $T: V\to V$ linear operator ⇒ $\exists$ invariant subspace of dim. 1 or 2 $W\subset V$

+ if T is orthogonal for $\langle\cdot,\cdot\rangle$ then it maps $W^\perp$ to $(T(W))^\perp = W^\perp$.

⇒ Thm: If $T: V\to V$ is an orthogonal operator on a finite dim. inner product space, then V decomposes into a direct sum of orthogonal invariant subspaces $V=\bigoplus V_i$, $V_i\perp V_j\ \forall i\neq j$, (ie. $V_i\subset V_j^\perp$) $T(V_i)=V_i$, of $\dim V_i\in\{1,2\}$.

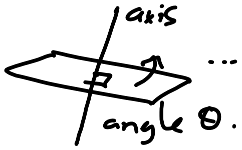
and if $\dim V_i = 1$ then $T_{|V_i}=\pm I$

if $\dim V_i = 2$ then $T_{|V_i}$ is either a rotation or reflection

(in latter case, can further decompose into ± 1 eigenspaces, so can replace reflections by 1-dim. blocks)

This gives a very nice way to think about an individual transformation as built from reflections and rotations on individual subspaces, but it's pretty useless for understanding the composition of two orthogonal transformations (whose invariant subspaces have no reason to coincide)

(Ex: rotations in $\mathbb{R}^3$:



... formula for the product of two rotations?)

Now on to the analogue of all this for complex vector spaces: Hermitian inner products

As previously noted, a bilinear form on a complex vector space $V\times V\to\mathbb{C}$ can't be definite positive, since $b(iv,iv)=-b(v,v)$. Solution: abandon $\mathbb{C}$-linearity in one of the two variables, and only require "conjugate linear"

Def: A Hermitian form on a complex vector space V is $H: V\times V\to\mathbb{C}$ st.

H is sesquilinear:

- $H(u+v,w)=H(u,w)+H(v,w)$, $H(u,v+w)=H(u,v)+H(u,w)$.
- $H(u,\lambda v)=\lambda H(u,v)$, however $H(\lambda u,v)=\bar\lambda H(u,v)$ ↳ conjugate: $\overline{a+ib}=a-ib$.

+ H conjugate-symmetric:

- $H(u,v)=\overline{H(v,u)}$.

Conjugate symmetry ⇒ $H(u,u)\in\mathbb{R}$.

Def: A Hermitian inner product is a positive definite (conjugate-symmetric) Hermitian form.

↳ ie. $H(u,u)\geq 0\ \forall u$, $H(u,u)=0 \Leftrightarrow u=0$.

Rmk: $\varphi_H : \begin{array}{l} V\to V^* \\ u\mapsto H(u,\cdot)\end{array}$ is now a complex antilinear map $V\to V^*$! $(\varphi(\lambda u)=\bar\lambda\varphi(u))$.