Group actions:

(Artin §6.7–6.9)

Def: An action of a group G on a set S is a homomorphism $\rho: G \to \mathrm{Perm}(S)$.
equivalently, we have a map $G \times S \longrightarrow S$, $(g,s) \mapsto g\cdot s$ s.t. $e\cdot s = s$ $\forall s \in S$, $(gh)\cdot s = g\cdot(h\cdot s)$

This generalizes the idea of groups as symmetries of geometric objects.
Understanding what sets a group G acts on (& in what way) gives info about G!

Def: An action is faithful if ρ is injective
(otherwise, the group that "really" acts on S is $G/\ker\rho$ ...)

Def: The orbit of $s \in S$ under G is $\mathcal{O}_s = G\cdot s = \{g\cdot s \,/\, g \in G\} \subset S$.

Observe: $t \in \mathcal{O}_s \iff \exists g \in G$ st. $g\cdot s = t$, and then $s = g^{-1}\cdot t \in \mathcal{O}_t$.
So: the orbits of the G-action form a partition of $S = \bigsqcup \mathcal{O}_s$.

Equivalently: $s \sim t \iff \exists g \in G$ st. $g\cdot s = t$ is an equivalence relation:
- $s \sim s$ since $e\cdot s = s$
- $s \sim t \Rightarrow \exists g,\ g\cdot s = t$, then $t = g^{-1}\cdot s$ so $t \sim s$.
- $s \sim t$ and $t \sim u \Rightarrow \exists g,\ g\cdot s = t$; $h,\ h\cdot t = u$ then $(hg)\cdot s = h\cdot(g\cdot s) = u$ hence $s \sim u$.

Orbits are the equivalence classes of this relation.

Def: An action is transitive if there is only one orbit.
i.e. $\forall s,t \in S$, $\exists g$ st. $g\cdot s = t$.

Note: Given any G-action on S, by restriction we get a G-action separately on each orbit. Each of these is transitive (by def^n), so we can break up any group action into a disjoint union of transitive actions!

Def:
- The stabilizer of $s \in S$ is $\mathrm{Stab}(s) = \{g \in G \mid g\cdot s = s\}$.
  This is a subgroup of G!
- The fixed points of $g \in G$ are the subset $S^g := \{s \in S \mid g\cdot s = s\}$.

$\star$ If $s' = g\cdot s$ then $\mathrm{Stab}(s') = g\,\mathrm{Stab}(s)\,g^{-1}$. So: elements in same orbit have conjugate stabilizers.

Pf: $h\cdot s = s \Rightarrow (ghg^{-1})gs = g(hs) = gs$, so $g\,\mathrm{Stab}(s)\,g^{-1} \subset \mathrm{Stab}(s')$.
conversely, same argument for $s = g^{-1}s' \Rightarrow g^{-1}\mathrm{Stab}(s')g \subset \mathrm{Stab}(s)$ hence equality).

* Example: given a subgroup $H \subset G$, we have a set $G/H = \{\text{cosets } aH\}$.
To avoid notation confusion, write $[H]$, $[aH]$, ... for elements of G/H.
G acts on G/H by left multiplication: $g\cdot[aH] = [gaH]$. This action is transitive (ba^{-1} maps $[aH]$ to $[bH]$). $\mathrm{Stab}([H]) = H$ itself, and $\mathrm{Stab}([aH]) = aHa^{-1}$.

Claim: this is what a general group action looks like when restricted to an orbit!

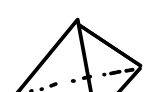
> If G acts on a set S, given $s \in S$, let $H = \mathrm{stab}(s) \subset G$. Then
> $\varepsilon : G/H \to \mathcal{O}_s$, $[aH] \mapsto a\cdot s$ is a bijection, and equivariant, ie. intertwines the G-actions:
> $\varepsilon(g\cdot[aH]) = g\cdot \varepsilon([aH])$
> action on G/H action on $\mathcal{O}_s \subset S$.

* well-def^d: if $a' = ah \in aH$ then $a'\cdot s = a\cdot h\cdot s = a\cdot s$ ✓
* surjective by def^n of orbit $\mathcal{O}_s = \{g\cdot s \,/\, g \in G\}$
* injective: $a'\cdot s = a\cdot s \Leftrightarrow a^{-1}(a'\cdot s) = a^{-1}(a\cdot s) = s \Leftrightarrow a^{-1}a' \in \mathrm{Stab}(s) = H \Leftrightarrow a' \in aH$.

Ie. the action of G on the orbit $\mathcal{O}_s$ is the same as on $G/\mathrm{Stab}(s)$, and the action of G on S is obtained as a disjoint union over orbits.

Corollary: || If G and S are finite, $|\mathcal{O}_s| = \dfrac{|G|}{|\mathrm{Stab}(s)|}$, and $|S| = \sum |\mathcal{O}_s|$.
since $\mathcal{O}_s \simeq G/\mathrm{Stab}(s)$ since $S = \coprod$ orbits

Ex: Let G = group of rotational symmetries of a tetrahedron



acting on S = set of faces ($|S| = 4$).
The action is transitive, ie. only one orbit, $|\mathcal{O}| = |S| = 4$
The stabilizer of an element $s \in S$ = rotations mapping a face to itself
$\Rightarrow |\mathrm{Stab}(s)| = 3$, and so we find $|G| = |\mathcal{O}_s|\cdot|\mathrm{Stab}(s)| = 4\cdot 3 = 12$.
(In fact $G \simeq A_4 \subset S_4$: id; 8 elts of order 3

$\leftrightarrow$ 3-cycles, 120°
3 elts of order 2 180°

$\leftrightarrow (12)(34)$ etc.)

Burnside's lemma = formula to count orbits of a group action.
Let G finite group acting on a finite set S, consider

$\Sigma = \{(g,s) \in G \times S \mid g \cdot s = s\}$. Two ways of calculating $|\Sigma|$:

→ as a sum over G: $|\Sigma| = \sum_{g \in G} |S^g|$ (recall: fixed points of g).

→ as a sum over S: $|\Sigma| = \sum_{s \in S} |\mathrm{Stab}(s)|$

But, since all elements in an orbit $\mathcal{O}$ have conjugate stabilizers, of size $|\mathrm{stab}(s)| = |G|/|\mathcal{O}|$ as seen above ($\mathcal{O}_s \simeq G/\mathrm{Stab}(s)$), we can rewrite this by grouping over orbits:

$$|\Sigma| = \sum_{s \in S} |\mathrm{Stab}(s)| = \sum_{\mathcal{O} \text{ orbit}} \left(|\mathcal{O}| \cdot |\mathrm{Stab}|\right) = \sum_{\mathcal{O} \text{ orbit}} |\mathcal{O}| \cdot \frac{|G|}{|\mathcal{O}|} = |G| \cdot (\#\text{orbits})!$$

Hence: <u>Burnside's lemma</u>: $\displaystyle \# \text{orbits} = \frac{1}{|G|} \sum_{g \in G} |S^g|$

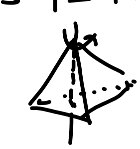
(the average # of fixed pts of elts of G).

<u>Ex</u>: how many ways to color faces of a tetrahedron with 3 colors, up to symmetries?

$S = \{\text{colorings of the faces}\} = \{\text{colors}\}^{\{\text{faces}\}}$, $|S| = 3^4 = 81$.

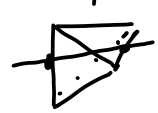
$G = A_4$ rotations of the tetrahedron.

- e = identity: $|S^g| = |S| = 81$.
- $120°$ rotation g (8 such g's)



S^g = 3 sides have same color $\Rightarrow |S^g| = 3 \times 3 = 9$.
sides bottom

- $180°$ rotation (3 such g's)



$|S^g| = 3 \times 3 = 9$ (front/back one color, top/bottom — " —)

$$\Rightarrow n = \frac{1}{|G|} \sum_{g \in G} |S^g| = \frac{1}{12}(81 + 11 \cdot 9) = \frac{180}{12} = 15.$$

(Could get this answer by different means... but e.g. coloring <u>edges</u> of tetrahedron would get harder w/out Burnside. Here: $\frac{1}{12}(3^6 + 8 \cdot 9 + 3 \cdot 3^4) = 87$.)

<u>Actions of G on itself</u>: (Artin §7.1–7.2)

1) G acts on itself by left multiplication, $g \cdot h = gh$.
This is transitive, with $\mathrm{Stab}(h) = \{e\}$ $\forall h \in G$, fixed points $= \emptyset$ $\forall g \neq e$.
It's faithful, $G \hookrightarrow \mathrm{Perm}(G)$. So we get

<u>Thm</u>: every finite group G is isomorphic to a subgroup of S_n, $n = |G|$.

This is not very useful for understanding G, however. More useful action:

2) G acts on itself by <u>conjugation</u>: g acts by $h \mapsto ghg^{-1}$.

We've seen that this does define a group homomorphism $G \to \mathrm{Aut}(G) \subset \mathrm{Perm}(G)$, so it is indeed an action. Now we have a more interesting structure:

the orbits of this action are <u>conjugacy classes</u> in G, and the stabilizer of an element $h \in G$ is $\mathrm{Stab}(h) = \{g \in G \,/\, gh = hg\}$ $(ghg^{-1} = h \Leftrightarrow gh = hg)$.

the subgroup of elements which commute with h. This is called the <u>centralizer</u> of h, $Z(h) \subset G$. Note $\bigcap_{h \in G} Z(h) = Z(G)$ the center of G is the kernel of the action (ie. the subgroup of elements which act trivially)

So: the action is trivial when G is abelian; faithful iff $Z(G) = \{e\}$.

* How does this help?

  - The conjugacy classes form a partition of G, so $\boxed{|G| = \sum_{\substack{C \subset G \\ \text{conj. class}}} |C|,}$ $(*)$

    For each conjugacy class, $|C_h| = \dfrac{|G|}{|Z(h)|}$ divides $|G|$.

    Moreover $|C_e| = 1$ for the identity element, and $|C_h| = 1$ iff $h \in Z(G)$.

    $(*)$ is called the <u>class equation</u> of the group G.

    This is extremely useful. For example:

<u>Theorem</u>: ‖ If $|G| = p^2$ for p prime, then G must be abelian.

<u>Proof</u>:
- conjugacy classes have order $|C| \in \{1, p, p^2\}$, and $\sum |C| = p^2$.
  Thus, the number of conjugacy classes s.t. $|C| = 1$, ie. of central elements of G, must be a multiple of p. Hence $p \mid |Z(G)|$.
- $Z(G)$ is a subgroup of G, so $|Z(G)|$ divides p^2: it's p or p^2.
  If $|Z(G)| = p^2$ then G is abelian!
- Now assume $|Z(G)| = p$, and let $g \notin Z(G)$. Then g commutes with itself and with $Z(G)$, so $Z(g) \supset Z(G) \cup \{g\}$, hence $|Z(g)| > p$. But $Z(g)$ is a subgroup of G, so $|Z(g)| \mid p^2$.
  This implies $Z(g) = G$, ie. g commutes with all elements of G, ie. $g \in Z(G)$, contradiction. So $Z(G) = G$, G is abelian. □

(Hence the only groups of order p^2 up to iso are $\mathbb{Z}/p^2$ and $\mathbb{Z}/p \times \mathbb{Z}/p$).

- <u>Proposition</u>: ‖ There are exactly 5 groups of order 8 up to isom.

We know the 3 abelian ones: $\mathbb{Z}/8$, $\mathbb{Z}/2 \times \mathbb{Z}/4$, $(\mathbb{Z}/2)^3$.

We know D_4 = symmetries of the square.

mult by -1 flips signs

Finally: quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ with $i^2 = j^2 = k^2 = -1$,
$ij = k$, $jk = i$, $ki = j$

Two ways to show there's only two nonabelian groups of order 8:

- "by hand" - see HW hint: if $|G| = 8$ and G not abelian.
  Step 1: a group where every element has $g^2 = 1$ must be abelian, so there must be an element a of order 4 (order 8 would make $G \simeq \mathbb{Z}/8$)
  Step 2: the order 4 subgroup generated by a is normal. Work out possibilities for mult. by an element b such that $ab \neq ba$.

- using conjugacy and class equation:
  Step 1: class equation $8 = \sum |C|$, $|C| \in \{1, 2, 4, 8\}$, $|C_e| = 1$
  $\Rightarrow Z(G) = \{g \mid |C_g| = 1\}$ has order 2, 4, or 8. $8 \Rightarrow G$ abelian.
  4 is impossible by same argument as for p^2 above. So $|Z(G)| = 2$.
  Step 2: if $g \notin Z(G)$ then $Z(g) \subsetneq G$, but $Z(G) \cup \{g\} \subset Z(g)$. So $|Z(g)| = 4$, and $|C_g| = 2$. Hence class equation is $8 = \underbrace{1 + 1}_{\text{e and the other central element}} + \underbrace{2 + 2 + 2}_{\text{3 other conj. classes}}$

  Then work out the possibilities!