<u>Recall</u>: • an <u>action</u> of G on a set S is a map $G \times S \to S$, $(g, s) \mapsto g.s$ st $(gh).s = g.(h.s)$, $e.s = s$.

or equivalently a homomorphism $G \to \mathrm{Perm}(S)$.

• given $s \in S$, the <u>orbit</u> $\mathcal{O}_s = \{g.s \,/\, g \in G\}$ and the <u>stabilizer</u> subgroup $\mathrm{Stab}(s) = \{g \in G \,/\, g.s = s\}$ are related by $G/\mathrm{Stab}(s) \simeq \mathcal{O}_s$, $[a\,\mathrm{Stab}] \mapsto a.s$.

$(g.s = g'.s$ iff $g^{-1}g' \in \mathrm{Stab}(s))$

<u>Today</u>: Use these ideas to classify <u>finite subgroups of $SO(3)$</u> $= \{$rotations of $\mathbb{R}^3\}$.

(& hence classify regular polyhedra, as well).

<u>Recall</u>: $(V, \langle\cdot,\cdot\rangle)$ inner product space $\rightsquigarrow$ orthogonal group

$$O(V) = \{T \in GL(V) \,/\, \langle Tu, Tv\rangle = \langle u, v\rangle\ \forall u, v \in V\}.$$

Elements of $O(V)$ have $\det = \pm 1$, and $SO(V) = \{T \in O(V) \,/\, \det T = 1\}$.

("the connected component of Id in $O(V)$").

We've seen $T \in O(V) \Rightarrow \exists$ decomposition $V = \bigoplus V_i$, $V_i \perp V_j$, $\dim V_i \in \{1, 2\}$, $T(V_i) = V_i$.

(uses: $\exists$ inv't subspace + if W is invariant then so is $W^\perp$).

if $\dim V_i = 1$, $T_{|V_i} = \pm 1$, if $\dim V_i = 2$, $T_{|V_i}$ = rotation.

• In dimension 3, either $T \sim \begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{pmatrix}$ or $T \sim \left(\begin{array}{c|c} \pm 1 & \\ \hline & \text{rotation} \end{array}\right)$

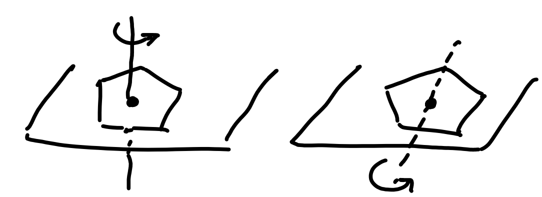
The condition $\det(T) = 1$ narrows it down to Id, $\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$, $\left(\begin{array}{c|c} 1 & \\ \hline & \text{rotation} \end{array}\right)$

$\Rightarrow$ every element of $SO(3)$ is a <u>rotation</u>; if $T \neq \mathrm{id}$, it has an axis (the $+1$-eigenspace = a line) and rotates by some angle in plane $\perp$ axis.

• Given a subset $\Sigma \subset \mathbb{R}^3$, can look at the symmetry group $\{T \in SO(3) \,/\, T(\Sigma) = \Sigma\}$.

It could be infinite (eg. if Σ is a circle in a plane, all rotations with axis $\perp$ plane will be symmetries), or it could be finite.

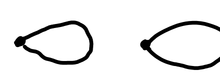
<u>Ex</u>: Σ = regular n-gon in a plane (centered at origin)



$\rightarrow$ n rotations (axis $\perp$ plane, angle $\frac{2\pi k}{n}$)

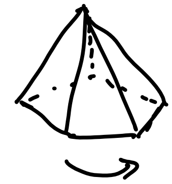
$\rightarrow$ n flips = rotation by π with axis $\subset$ plane

$\Rightarrow$ this is isomorphic to D_n.

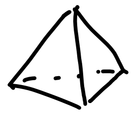
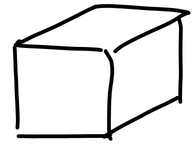


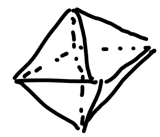
(special cases $n=1$ $n=2$

Ex: to only keep $\mathbb{Z}/n \subset D_n$ in the above example, consider a cone on a regular n-gon in a plane:



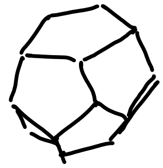
Ex: symmetries of regular polyhedra:

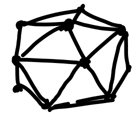
tetrahedron  , cube 

octahedron 

these have the same symmetries, by duality vertices $\leftrightarrow$ centers of faces.

(or: $P^* = \{v \in \mathbb{R}^3 / \langle v, u\rangle \leq 1 \ \forall u \in P\}$)

dodecahedron  (12 pentagonal faces)

icosahedron  (20 triangular faces)

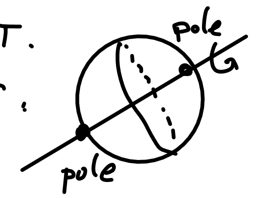
duals, have same symmetries.

These give respectively A_4 (seen last time), S_4 (HW2!), A_5.

action on vertices or faces of  ; action on the 4 diagonals of  ; action on what?

Theorem: This is the complete list of finite subgroups of $SO(3)$:
$\mathbb{Z}/n$, D_n, tetrahedron (A_4), cube (S_4), icosahedron (A_5)

★ The key observation is that every $T \in SO(3)$, $T \neq id$ is a rotation about some axis, hence fixes exactly two unit vectors $\pm v$, called the <u>poles</u> of T.



For $G < SO(3)$ finite, let P = the set of all poles of elements of G:

$$P = \{v \in \mathbb{R}^3 \mid \|v\| = 1 \text{ and } \exists g \in G,\ g \neq 1 \text{ st. } gv = v\}.$$

Now, if v is a pole of $g \in G$, and given any $h \in G$, $h(v)$ is a pole of $hgh^{-1} \in G$ (since $hgh^{-1} \cdot hv = hgv = hv$).

So G acts on P! This is the key to understanding the group G.

★ **Ex:** in the case of symmetry groups of regular polyhedra:

$$P = \{\text{vertices}\} \cup \{\text{centers of faces}\} \cup \{\text{midpoints of edges}\}.$$

These form 3 different orbits for the action of G on P.
(namely, G acts separately on vertices, on faces, and on edges; for a regular polyhedron each of these actions is transitive).

★ The next observation is that for $p \in P$, $\mathrm{Stab}(p)$ consists of rotations with axis $\pm p$!

These form an abelian, in fact cyclic subgroup of G

So: $\mathrm{Stab}(p) \cong \mathbb{Z}/r_p$, for some integer $r_p > 1$ (since p is a pole of some element of G, $\mathrm{Stab}(p)$ must be nontrivial)

(ie. angles of rotations through p form a finite subgroup of $\mathbb{R}/2\pi\mathbb{Z}$, must be all multiples of $2\pi/r_p$). With this understood, the proof of the theorem is a counting argument.

Pf: Let $G \subset SO(3)$ a nontrivial finite subgroup, $\mathcal{P}$ the set of poles as above.

Let $\Sigma = \{(g,p) \in G \times \mathcal{P} \,/\, g \neq e,\ g(p) = p\}$ (ie. p is a pole of g)

For each element of $G - \{e\}$, there are exactly 2 poles. So $|\Sigma| = 2|G| - 2$.

For each element $p \in \mathcal{P}$, there are $r_p - 1$ rotations in $G - \{e\}$ fixing p.

So: $$|\Sigma| = 2|G| - 2 = \sum_{p \in \mathcal{P}} (r_p - 1).$$

Now, the elements $p \in \mathcal{O}$ of an orbit of G have conjugate stabilizers ($\mathrm{Stab}\, g(p) = g\, \mathrm{Stab}(p)\, g^{-1}$), hence same r_p's: $r_{gp} = r_p$.

Thus: $$2|G| - 2 = \sum_{\mathcal{O}_i \text{ orbit}} |\mathcal{O}_i| (r_i - 1).$$ where $r_i = r_p$ for $p \in \mathcal{O}_i$.

Now remember orbit/stabilizer: $|\mathcal{O}_i| = \dfrac{|G|}{|\mathrm{Stab}|} = \dfrac{|G|}{r_i}$, so

$$2|G| - 2 = \sum_{\mathcal{O}_i \text{ orbit}} \frac{|G|}{r_i}(r_i - 1), \quad \text{ie.} \quad 2 - \frac{2}{|G|} = \sum_{\text{orbits}} 1 - \frac{1}{r_i}$$

The rhs gets ≥ 2 quickly if there are too many orbits !!
each term is $\geq \frac{1}{2}$ since $r_i \geq 2$, hence #orbits ≤ 3.

We now analyze each case based on the number of orbits.

- 1 orbit: impossible, lhs ≥ 1 (since $|G| \geq 2$) vs. rhs < 1.
- 2 orbits: $2 - \dfrac{2}{|G|} = 1 - \dfrac{1}{r_1} + 1 - \dfrac{1}{r_2}$, ie. $\dfrac{2}{|G|} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$.

  Since each $r_i = |\mathrm{Stab}\, p|$ divides $|G|$, we must have $r_1 = r_2 = |G|$.
  Hence $\mathrm{Stab} = G$, ie. there are 2 poles $\pm p$, each fixed under all of G, and $G = \mathrm{Stab}(p) = \mathbb{Z}/r$ cyclic subgroup of $\frac{2\pi k}{r}$ rotations with axis $\pm p$.
- 3 orbits: $2 - \dfrac{2}{|G|} = 3 - \dfrac{1}{r_1} - \dfrac{1}{r_2} - \dfrac{1}{r_3}$. Assume $2 \leq r_1 \leq r_2 \leq r_3$.

Observe: $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = 1 + \dfrac{2}{|G|} > 1 \Rightarrow$ necess. $r_1 = 2$ (else: $\sum \frac{1}{r_i} \leq \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$)
$r_2 \leq 3$ (else $\sum \frac{1}{r_i} \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$)

(a) If $r_2 = 2$: then $2 - \frac{2}{|G|} = 3 - \frac{1}{2} - \frac{1}{2} - \frac{1}{r_3} \Rightarrow r_3 = \frac{|G|}{2}$.

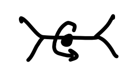
Thus, $|O_3| = \frac{|G|}{r_3} = 2$, two poles form an orbit. These poles are necessarily $\pm p$, and

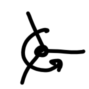
- half of G = rotations $\frac{2\pi k}{r_3}$ about $\pm p$ (the stabilizer of $\pm p$)
- the other half of G = rotations by 180° (since these poles have $r=2$) & swapping $p \leftrightarrow -p$ (since G preserves the orbit $\{\pm p\}$)

$\Rightarrow G$ = dihedral group.

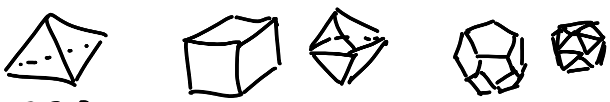
(b) If $r_2 = 3$: $\sum \frac{1}{r_i} > 1 \Rightarrow r_3 \in \{3,4,5\}$.

These 3 cases give the tetrahedron, cube, and icosahedron. □

<u>NB:</u> for regular polyhedra: poles at midpoints of edges have $r=2$ 

poles at vertices: r = # faces meeting 

poles at center of face: r = # edges of the face 



2,3,3 2,3,4 2,3,5

$\frac{2}{|G|} = \sum \frac{1}{r_i} - 1 \Rightarrow |G| = 12, 24, 60.$

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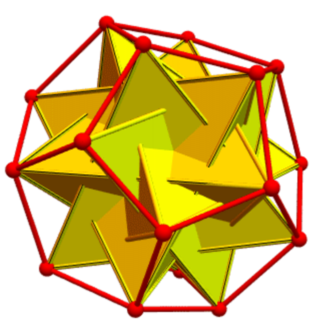
* What is the 5-element set that symmetries of the dodecahedron act on?



(Image by Greg Egan)

<u>Ans:</u> the 20 vertices of a dodecahedron can be partitioned into 5 sets of 4 forming regular tetrahedra (in 2 different ways which are mirror images, but not related by a rotation).

A rotation of the dodecahedron then permutes the 5 tetrahedra.

- Rotations / center of faces $\leftrightarrow$ 5-cycles (24 of them)
- Rotations / vertices $\leftrightarrow$ 3-cycles (123) etc. (20 of them)
- Half-rotations / midpoints of edges $\leftrightarrow$ (12)(34) etc. (15 of them)