Lec. 1
Artin ch. 2

- A **group** $(G,\cdot)$ is a set with an operation $\cdot : G\times G\to G$ st. (1) $\exists e\in G$ **identity** s.t. $eg=ge=g\ \forall g\in G$, (2) $\forall g\in G\ \exists$ **inverse** $g^{-1}\in G$ st. $gg^{-1}=g^{-1}g=e$, (3) **associativity** $(ab)c=a(bc)\ \forall a,b,c\in G$.
- A group is **abelian** if $\cdot$ is commutative ($ab=ba\ \forall a,b\in G$)
- **Ex**: $(\mathbb{Z},+)$, $(\mathbb{Z}/n,+)$, $(\mathbb{C}^*,\times)$, symmetric group S_n; $GL_n(\mathbb{R})$ etc.; **products** $G\times H$, $\mathbb{Z}^n$, ...
- like sets, groups can be finite ($\mathbb{Z}/n$, S_n, ...), countable ($\mathbb{Z}$, $\mathbb{Z}^n$, $\mathbb{Q}$, ...), uncountable ($\mathbb{R}$...)

Lec. 2

- $H\subset G$ is a **subgroup** if $e\in H$, $a\in H\Rightarrow a^{-1}\in H$, $a,b\in H\Rightarrow ab\in H$. $|H|$ divides $|G|$.
  H, H' subgroups of $G\Rightarrow H\cap H'$ is a subgroup of G.
  All subgroups of $(\mathbb{Z},+)$ are $\mathbb{Z}n=\{mn\,/\,m\in\mathbb{Z}\}$ for some $n\geq 0$.
- A **homomorphism** $\varphi: G\to H$ is a map st. $\varphi(ab)=\varphi(a)\varphi(b)\ \forall a,b\in G$. ($\Rightarrow \varphi(a^{-1})=\varphi(a)^{-1}$)
  isomorphism = bijective homomorphism, automorphism = isom $G\xrightarrow{\sim} G$. $(\mathrm{Aut}(G),\circ)$ is a group.
- The **kernel** of $\varphi: G\to H$: $\mathrm{Ker}(\varphi)=\{g\in G\,/\,\varphi(g)=e_H\}$ subgroup of G. φ injective $\Leftrightarrow \mathrm{Ker}\,\varphi=\{e\}$
  The **image** of φ: $\mathrm{Im}(\varphi)=\{\varphi(g)\,/\,g\in G\}\subset H$ subgroup of H. φ surjective $\Leftrightarrow \mathrm{Im}\,\varphi=H$.
- Given $a\in G$, $\varphi: \begin{array}{l}\mathbb{Z}\to G\\ k\mapsto a^k\end{array}$ is a homomorphism with $\mathrm{im}(\varphi)=\langle a\rangle$ subgp generated by a.
  $\mathrm{Ker}(\varphi)=\mathbb{Z}n$ where $n=$ order of $a=\min\{n>0 \text{ st. } a^n=e\}$.
  Hence the **cyclic** group $\langle a\rangle$ is $\simeq \mathbb{Z}/n$ if a has order n, $\simeq\mathbb{Z}$ if infinite order.
  ($a_1,\dots,a_k$ generate G if every element of G is a product of a_i and their inverses).

Lec. 3

- A subgroup $H\subset G$ determines an **equivalence relation** (axioms: $a\sim a$; $a\sim b\Leftrightarrow b\sim a$; $\left.\begin{array}{l}a\sim b\\ b\sim c\end{array}\right\}\Rightarrow a\sim c$)
  $a\sim b$ iff $a^{-1}b\in H$, whose equivalence classes are the (left) **cosets** $aH=\{ah\,/\,h\in H\}$.
  The quotient set: $G/H=\{\text{cosets } aH\}$. The index of H: $(G:H)=|G/H|=\dfrac{|G|}{|H|}$ if finite.

Lec. 4

- If G is finite: $H\subset G$ subgroup $\Rightarrow |H|$ divides $|G|$; $a\in G\Rightarrow \mathrm{ord}(a)\,|\,|G|$; $|G|=p$ prime $\Rightarrow G\simeq\mathbb{Z}/p$.
- A subgroup $H\subset G$ is **normal** $\Leftrightarrow \underset{\text{(left cosets = right cosets)}}{aH=Ha\ \forall a\in G}\Leftrightarrow \underset{\text{(conjugate subgroups)}}{aHa^{-1}=H\ \forall a\in G}$.
- The operation $(aH).(bH)=abH$ makes G/H a **group** iff H is a **normal subgroup**.
- $\forall \varphi: G\to H$ homomorphism, $\mathrm{Ker}(\varphi)=K$ is a normal subgroup of G, and $\mathrm{Im}(\varphi)\simeq G/K$.
  If φ is surjective, we have an **exact sequence** $\{1\}\to K\xrightarrow[\text{injective}]{i} G\xrightarrow[\text{surjective}]{\varphi} H\to\{1\}$ $\mathrm{Im}(i)=\mathrm{Ker}(\varphi)$.
  Ex. $\{1\}\to \underset{\text{normal}}{H}\hookrightarrow G\to G/H\to\{1\}$; $0\to\mathbb{Z}/m\to\mathbb{Z}/mn\to\mathbb{Z}/n\to 0$ ($\mathbb{Z}/mn\simeq\mathbb{Z}/m\times\mathbb{Z}/n$ iff $\gcd(m,n)=1$)
  $\{e\}\to\mathbb{Z}/3\to S_3\xrightarrow{\text{sign}}\mathbb{Z}/2\to\{e\}$
  A homomorphism $G\xrightarrow{\varphi}H$ factors through $G\to G/K\xrightarrow{\bar\varphi}H$ iff $K\subset\mathrm{Ker}\,\varphi$
- G is **simple** if its only normal subgroups are $\{e\}$ and itself. **Ex**: $\mathbb{Z}/p$ p prime; A_n $n\geq 5$.
- **Ex**: the **center** $Z(G)=\{z\in G\,/\,zg=gz\ \forall g\in G\}$ is a normal subgroup (abelian: $zz'=z'z$)

Lec. 5

- **Ex**: the **commutator** subgroup $[G,G]=\{\prod_{\text{finite}}[a_i,b_i]\}$, where $[a,b]=aba^{-1}b^{-1}$, is normal, and $G/[G,G]=\mathrm{Ab}(G)$ (abelianization) largest abelian quotient of G. $\forall G\xrightarrow{\varphi}H$, H abelian factors $G\to\mathrm{Ab}(G)\xrightarrow{\bar\varphi}H$.

Lec. 19
Artin 14.7

- Every finitely generated abelian group is $\simeq\mathbb{Z}^r\times\mathbb{Z}/n_1\times\dots\times\mathbb{Z}/n_k$ for some $r,n_1,\dots,n_k$.

Artin 6.7–6.12 • <u>Group actions</u>: G-action on set S: $G\times S \to S$, $(g,s)\mapsto g\cdot s$ st. $e.s=s\ \forall s\in S$, $(gh).s=g.(h.s)$ ($\Leftrightarrow$ homom. $\rho: G\to \mathrm{Perm}(S)$) ②

<u>faithful</u> if ρ injective; <u>transitive</u> if $\forall s,t\in S\ \exists g$ st. $gs=t$ (ie: 1 orbit)

Lec.20 • The <u>orbit</u> of $s\in S$ is $O_s = G\cdot s = \{g\cdot s \mid g\in G\}$. These form a partition $S=\coprod$ orbits.

The <u>stabilizer</u> of s is $\mathrm{Stab}(s)=\{g\in G \mid g.s=s\}$ subgroup of G.

Elements in same orbit have conjugate stabilizer subgroups $\mathrm{Stab}(g.s)=g\,\mathrm{Stab}(s)\,g^{-1}\subset G$.

Orbit-stabilizer: if $H=\mathrm{Stab}(s)$, then $G/H \simeq O_s$, $gH\mapsto g.s$ bijection, in particular $|O_s|\cdot|\mathrm{Stab}\,s|=|G|$.

• Burnside's lemma (G, S finite): let $S^g=\{s\in S \mid gs=s\}$ fixed points of $g\in G$, then $\#\text{orbits} = \frac{1}{|G|}\sum_{g\in G}|S^g|$

Artin ch.7 • G acts on itself by left multiplication. This gives $G\hookrightarrow \mathrm{Perm}(G)$, hence: every finite group G is isomorphic to a subgroup of S_n, $n=|G|$.

• G acts on itself by <u>conjugation</u>: g acts by $h\mapsto ghg^{-1}$.

orbits = conjugacy classes; $\mathrm{Stab}(h)=\{g\in G \mid gh=hg\}=Z(h)$ <u>centralizer</u> of h.

Hence: $|G|=\sum_{\substack{C\subset G\\ \text{conj. classes}}}|C|$, where for each conj. class $|C_h|=\frac{|G|}{|Z(h)|}$ divides $|G|$. (<u>class eq^n</u> of G)

• For p-groups ($|G|=p^k$), the class equation $\Rightarrow |Z(G)|\geq p$ (number of conj classes of size 1)

Hence: • $|G|=p^2$, p prime $\Rightarrow G$ is abelian ($\simeq \mathbb{Z}/p\times\mathbb{Z}/p$ or $\mathbb{Z}/p^2$)

• 5 isom. classes of groups of order 8: $\mathbb{Z}/8$, $\mathbb{Z}/4\times\mathbb{Z}/2$, $(\mathbb{Z}/2)^3$, D_4, quaternion group.

Lec.21: • $G\subset SO(3)$ finite subgp $\Rightarrow$ by considering the action of G on its poles (unit vectors along rotation axes), $G\simeq$ one of $\mathbb{Z}/n$, D_n (regular n-gon), A_4 (tetrahedron), S_4 (cube), A_5 (dodecahedron/icosahedron)

Lec.22: • The <u>symmetric group</u> S_n is generated by transpositions (ij), in fact by $s_i=(i\ i{+}1)$.

• $\forall\sigma\in S_n\ \exists$ unique decomp of σ as product of disjoint cycles $(a_1\dots a_k)$.

$\sigma,\tau\in S_n$ are in same conjugacy class iff they have the same cycle lengths.

• the <u>alternating group</u> $A_n=\mathrm{Ker}(\mathrm{sign}: S_n\to\mathbb{Z}/2)=\{$products of even # of transpositions$\}$

Lec.23: A conjugacy class in S_n which consists of even permutations is either 1 or 2 conj. classes in A_n; it splits into 2 iff the centralizer $Z(\sigma)\subset A_n$ ($\Leftrightarrow$ cycle lengths of σ are all odd & distinct).

• A_n is simple for $n\geq 5$ (A_4 isn't: $\{id, (ij)(kl)\}\simeq \mathbb{Z}/2\times\mathbb{Z}/2$ is normal in A_4 and S_4).

Lec.24: • <u>Sylow theorems</u>: $|G|=p^e m$, $p\nmid m$ $\Rightarrow$ a <u>Sylow p-subgroup</u> of G is a subgp. of order p^e.

Thm 1: $\forall p$ prime $\mid |G|$, G contains a Sylow p-subgroup. ($\to$ consequence: G contains an elt of order p)

Thm 2: all Sylow p-subgroups of G are conjugates of each other, and every subgroup of order p^k ($k\leq e$) is contained in a Sylow subgroup.

Thm 3: the number s_p of Sylow p-subgroups satisfies $s_p\equiv 1 \bmod p$ and $s_p \mid m=\frac{|G|}{p^e}$.

• If G contains subgroups N, H st. $N\cap H=\{e\}$ (eg because $\gcd(|N|,|H|)=1$) and $|G|=|N|\cdot|H|$, then $\forall g\in G\ \exists$ unique $n\in N$, $h\in H$ st. $g=nh$.

If N and H are <u>both</u> normal in G then $G\simeq N\times H$. If N is normal but not H, we have a semidirect product $N\rtimes_\varphi H$, $\varphi: H\to \mathrm{Aut}(N)$ given by conjugation inside G.

$$(n,h)\cdot(n',h')=(n\,\varphi(h)(n'),\ hh')$$

Lec. 25 • given $H \subset G$ (eg. p-Sylow), the number of conjugate subgroups $gHg^{-1} \subset G$ (eg. all p-Sylows) equals $|G/N(H)|$, $N(H)$ normalizer $= \{g \in G \mid gHg^{-1} = H\}$ (largest subgp of G st. H is normal inside N).

• Example: $|G| = 15 \Rightarrow$ Sylow subgroups of order 3 and 5 are normal ($s_3 = s_5 = 1$) $\Rightarrow G \simeq \mathbb{Z}/3 \times \mathbb{Z}/5$.
$|G| = 21 \Rightarrow s_3 \in \{1,7\}$, $s_7 = 1$, so either $G \simeq \mathbb{Z}/3 \times \mathbb{Z}/7$ or semidirect product $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$.

Lec. 26 $|G| = 12 \Rightarrow$ 1 or 3 2-Sylows, 1 or 4 3-Sylows, one of these is normal $\Rightarrow$ 5 isom. classes:
$\mathbb{Z}/4 \times \mathbb{Z}/3, (\mathbb{Z}/2)^2 \times \mathbb{Z}/3, A_4, D_6, \mathbb{Z}/3 \rtimes \mathbb{Z}/4$.

• The free group $F_n = \langle a_1 \dots a_n \rangle = \{$all reduced words $a_{i_1}^{m_1} \dots a_{i_k}^{m_k}\}$
(words in $a_i^{\pm 1}$ never simplify except $a_i a_i^{-1} = a_i^{-1} a_i = 1$)

• Any group G with n generators $g_1 \dots g_n$ is a quotient of F_n, via $\varphi: F_n \twoheadrightarrow G$, $a_i \mapsto g_i$
G is finitely presented if $\operatorname{Ker}(\varphi)$ is generated by a finite set $r_1, \dots, r_k$ & their conjugates
Write $G \simeq \langle g_1 \dots g_n \mid r_1 \dots r_k \rangle = F_n / \langle$normal subgp gen.d by conjugates of $r_j \rangle$.

• The Cayley graph of G w/ generators g_i: vertices = elements of G
edges: connect g to $g \cdot g_i$ $\forall g \in G$, $\forall g_i$

• A normal form for elements of $G = \langle g_1 \dots g_n \mid r_1 \dots r_k \rangle$ is a set of words in $g_1^{\pm 1} \dots g_n^{\pm 1}$ st. every element of G appears exactly once among these.

Lec. 27 • Ex: $S_n \simeq \langle s_1, \dots, s_{n-1} \mid s_i^2 = 1,\ s_i s_j = s_j s_i \ |i-j| \geq 2,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle$.
$SL_2(\mathbb{Z})$ is gen.d by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm I\} = \langle S, T \mid S^2, (ST)^3 \rangle$

Lec. 28 • A representation of G is a vector space V on which G acts by linear operators; ie. $\rho: G \to GL(V)$ homomorphism.

Artin ch. 10, Fulton-Harris ch. 1–2

• A subrepresentation is a subspace $W \subset V$ invariant under G: $g(W) = W$ $\forall g \in G$.
V is irreducible if has no nontrivial subrepresentations

• G finite, V finite dim. / $\mathbb{C}$: each $g: V \to V$ has finite order, $g^n = Id \Rightarrow$ diagonalizable, $\lambda_j = e^{2\pi i k/n}$

• if G is abelian, all operators $g: V \to V$ are simultaneously diagonalizable $\Rightarrow$ irred. rep.s are 1-dim.l.
These correspond to elements of the dual group $\hat{G} = \operatorname{Hom}(G, \mathbb{C}^*)$. (Note $\widehat{\mathbb{Z}/m}$ is $\simeq \mathbb{Z}/m$)

• a homomorphism of representations is a G-equivariant linear map, ie. $\varphi(gv) = g\varphi(v)$.

• V, W rep.s of $G \Rightarrow$ so are $V \oplus W$, $V \otimes W$ ($g: v \otimes w \mapsto gv \otimes gw$), V^* ($\ell \mapsto \ell \circ g^{-1}$),
$V^* \otimes W \simeq \operatorname{Hom}(V, W)$ ($\varphi \mapsto g \circ \varphi \circ g^{-1}$). ($\operatorname{Hom}_G(V,W) =$ invariant part $\operatorname{Hom}(V,W)^G$)

• Any $\mathbb{C}$-representation of a finite group G admits an invariant Hermitian inner product, with respect to which G acts by unitary operators.

Lec. 29 • V rep. of a finite group (over $\mathbb{C}$), $W \subset V$ invariant subspace $\Rightarrow \exists U \subset V$ invariant st. $V = U \oplus W$.
Hence: any $\mathbb{C}$-representation of a finite group decomposes into a direct sum of irreducibles.

• Schur's lemma: V, W irred. rep's of $G \Rightarrow$ any homom. $\varphi \in \operatorname{Hom}_G(V,W)$ is either zero or an isomorphism; and all iso's of an irred. rep. are multiples of id: $\operatorname{Hom}_G(V,V) = \mathbb{C} \cdot id_V$

• Ex: reps. of S_n: trivial rep $U = \mathbb{C}$, σ acts by id; alternating rep: $U' = \mathbb{C}$, σ acts by $(-1)^\sigma$.
standard rep. (dim. n-1): $V = \{(z_1, \dots, z_n) \mid \Sigma z_i = 0\} \subset \mathbb{C}^n$, σ acts by permuting coords: $e_i \mapsto e_{\sigma(i)}$.
U, U', V are the only irred. reps of S_3.

Lec. 30: • The key tool to study representations is the <u>character</u> $\chi_V : G \to \mathbb{C}$, $\chi_V(g) = tr(g : V \to V)$
(In terms of eigenvalues, $tr(g) = \sum \lambda_i$, and $tr(g^k) = \sum \lambda_i^k$, so χ_V recovers all symmetric polynomial expressions in the λ_i, hence the λ_i as unordered tuple).
$\chi_V : G \to \mathbb{C}$ is a class function, ie. $\chi_V(hgh^{-1}) = \chi_V(g)$.

• $\chi_{V \oplus W} = \chi_V + \chi_W$, $\chi_{V \otimes W} = \chi_V \chi_W$, $\chi_{V^*} = \overline{\chi_V}$, $\chi_{Hom(V,W)} = \overline{\chi_V}\, \chi_W$.

• for a permutation rep. (G acting on $S \rightsquigarrow G$ acts on V with basis $(e_s)_{s \in S}$, $g \cdot e_s = e_{g \cdot s}$)
$\chi(g) = \#\{s \in S / g \cdot s = s\} = |S^g|$.

• Character table of G = list, for each irred. rep. V_i, the value of χ_{V_i} on each conjugacy class.

• $\varphi = \frac{1}{|G|} \sum_{g \in G} g : V \to V$ projection onto $V^G = \{v \in V / gv = v \ \forall g\}$, so $\dim(V^G) = tr(\varphi) = \frac{1}{|G|} \sum_g \chi_V(g)$

Lec. 31 • $H(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha}(g) \beta(g)$ Hermitian inner product on $\mathbb{C}_{class}(G) = \{$class functions $G \to \mathbb{C}\}$
Then $\dim Hom_G(V, W) = H(\chi_V, \chi_W)$.

• The characters of the irreducible reps of G are an <u>orthonormal basis</u> of $(\mathbb{C}_{class}(G), H)$.
In particular the number of irred. reps = number of conjugacy classes

• The multiplicities a_i in the decomposition of a G-rep. W into irreducibles $W \simeq \bigoplus_i V_i^{\oplus a_i}$
are given by $a_i = \dim Hom_G(V_i, W) = H(\chi_{V_i}, \chi_W)$. Moreover, $H(\chi_W, \chi_W) = \sum a_i^2$.

• The regular repr. of G (= permutation rep. for G acting on itself by left multiplication)
contains each irred. rep. V_i with multiplicity $= \dim V_i$; therefore $|G| = \sum_i (\dim V_i)^2$.

Lec. 32–33 • These results allow us to find character tables of various groups (eg. S_4, A_4, S_5, A_5) by starting from known representations, considering tensor products, and using $H(\cdot, \cdot)$ pairings and orthogonality to find irreducible pieces & the missing irreducible reps.