<u>The group algebra</u> of a finite group G gives us another perspective on representations of G - not as immediately helpful for calculating characters & finding irreducibles, but conceptually important.

- <u>Def</u>: The <u>group algebra</u> of G is the vector space $\mathbb{C}G = \{ \sum_{g\in G} a_g e_g,\ a_g \in \mathbb{C}\}$, with the product $e_g \cdot e_h = e_{gh}$ (& extend by linearity) – this is a (noncommutative) ring.
$$\Big(\sum_g a_g e_g\Big)\Big(\sum_g b_g e_g\Big) = \sum_g \Big(\sum_h a_h b_{h^{-1}g}\Big) e_g \quad \text{(commutative iff } G \text{ is abelian)}$$

As a vector space this is the same as the regular rep^n; the new thing is the multiplication.

- An action of G on a vector space V (a representation) is a homom. $\rho: G \to GL(V)$ and extends by linearity to an algebra homomorphism (ie. linear map of vector spaces + multiplicative: ring homom.)
$$\mathbb{C}G \longrightarrow End(V) \text{ by mapping basis elements } e_g \mapsto \rho(g)$$
+ extend linearly: $\sum a_g e_g \mapsto \sum a_g \rho(g)$; to check it's compatible with multiplication, using (bi)linearity it's enough to check for basis elements: $e_g \cdot e_h = e_{gh} \mapsto \rho(gh) = \rho(g)\circ\rho(h)$. ✓

$\Rightarrow$ <u>Prop</u>: a G-representation is the same thing as a (left) <u>$\mathbb{C}G$-module</u>, namely a vector space V + an action $\mathbb{C}G \times V \to V$ given by a ring hom. $\mathbb{C}G \to End(V)$.

- <u>Ex</u>: the regular representation of G corresponds to $\mathbb{C}G$ as a module over itself! (operation of $\mathbb{C}G$ is left-multiplication)

Since we haven't learned much about rings and modules, we won't pursue this in depth. There is however one nice result worth seeing:

Given a finite group G, let $V_1, \dots, V_r$ be the irreducible rep^{ns} of G.
Each of these gives a ring homom. $\mathbb{C}G \to End(V_i)$; taking together, we get a map
$$\mathbb{C}G \longrightarrow \bigoplus_{i=1}^r End(V_i). \qquad \Big(\subsetneq End\Big(\bigoplus_{i=1}^r V_i\Big): \text{subring of block diagonal linear operators on } \bigoplus_{i=1}^r V_i\Big)$$
This map is again a ring homomorphism (product in $\mathbb{C}G \mapsto$ composition of End's).

<u>Prop</u>: If $V_1, \dots, V_r$ are the irred. reps of G, this map $\mathbb{C}G \longrightarrow \bigoplus_{i=1}^r End(V_i)$ is an <u>isomorphism</u> of rings.

<u>Pf</u>: We already know it's a homomorphism, so we just need to check it's bijective.
- the map is injective: assume $\sum a_g e_g \in \mathbb{C}G$ belongs to the kernel, then $\forall$ irred. rep. $\sum a_g \rho_{V_i}(g) = 0$, hence $\forall$ representation of G, $\sum a_g \rho(g) = 0$.
However, for the regular rep^n, the $\rho(g)$ are linearly indep^t ($\sum a_g \rho(g)$ maps e_1 to $\sum a_g e_g$.) so this implies $a_g = 0\ \forall g$.
- $\dim \mathbb{C}G = |G| = \sum (\dim V_i)^2 = \dim\big(\bigoplus End(V_i)\big)$, so an injective linear map is surjective. □

* In the ring $\bigoplus \mathrm{End}(V_i)$, as in any direct sum of rings, the projectors onto each summand $P_i = \begin{cases} \mathrm{Id} & \text{on } \mathrm{End}(V_i) \\ 0 & \text{on } \mathrm{End}(V_j),\ j\neq i \end{cases}$ are orthogonal idempotents: $P_i^2 = P_i$, $P_iP_j = 0$ for $i\neq j$.

Comparing with projection formulas: we've seen that $\forall$ rep$^{\text{n}}$ V, $\varphi_i = \frac{\dim V_i}{|G|}\sum_g \overline{\chi_{V_i}(g)}\, g : V\to V$ is the projection onto the V_i summands. This means: the idempotents of $\mathbb{C}G$ corresponding to the projectors P_i under the isom. are $\pi_i = \frac{\dim V_i}{|G|}\sum_{g\in G}\overline{\chi_{V_i}(g)}\, e_g \in \mathbb{C}G$.

(The identities $\pi_i^2 = \pi_i$, $\pi_i\pi_j = 0$ for $i\neq j$ recover, among other things, the orthonormality of χ_{V_i}!)

Given a $\mathbb{C}G$-module V, it has submodules $\pi_i V$ – these are the pieces of V consisting of the V_i summands in the decomposition of V.

<u>Real representations</u>: we've studied actions of finite groups on complex vector spaces, now we want to do the same for real ones.

* If V_0 is a representation of G over $\mathbb{R}$, then it has an <u>invariant inner product</u> $\langle\cdot,\cdot\rangle$. (start from any inner product $b(\cdot,\cdot)$, and let $\langle v_1,v_2\rangle = \frac{1}{|G|}\sum_{g\in G} b(gv_1, gv_2)$).
  $\leadsto$ the elements of G then act by orthogonal transformations (isometries).
* This implies <u>complete reducibility</u>: every representation$/\mathbb{R}$ splits into direct sum of irreducibles. (same pf as complex case: if $U_0\subset V_0$ invariant subspace (subrep.) then $V_0 = U_0\oplus U_0^{\perp}$)
* However, Schur's lemma <u>fails</u>.

  <u>Ex</u>: the action of $\mathbb{Z}/n$ on $\mathbb{R}^2$ by rotations, k acting by $\begin{pmatrix}\cos\frac{2\pi k}{n} & -\sin\frac{2\pi k}{n}\\ \sin\frac{2\pi k}{n} & \cos\frac{2\pi k}{n}\end{pmatrix}$ is irreducible as a rep. over $\mathbb{R}$. However this representation has automorphisms that aren't multiples of Id: any rotation of $\mathbb{R}^2$ is $\mathbb{Z}/n$-equivariant.

Therefore, a lot of the theory we've developed over $\mathbb{C}$ won't apply directly to rep's over $\mathbb{R}$. Instead, the key idea (just like when we discussed operators on $\mathbb{R}$-vect. spaces) is <u>complexification</u>

* We have a map $\{\text{real rep}^{\text{ns}}\ V_0\} \longrightarrow \{\text{complex rep}^{\text{ns}}\}$

$$V_0 \longmapsto V = V_0\otimes_{\mathbb{R}}\mathbb{C} = V_0\oplus iV_0.$$

(G acts by $g(v+iw) = gv + i\,gw$).

ie. given basis (e_j) of V_0, $e_j + 0i$ $(=e_j)$ basis of V; g acts by same matrix on V_0 and V.

<u>Def</u>: A complex rep. V of G is called <u>real</u> if there exists a rep. over $\mathbb{R}$, V_0, st. $V = V_0\otimes_{\mathbb{R}}\mathbb{C}$

Necessary condition: χ_V must take real values!

because: the matrix of $g: V\to V$ in suitable basis has real entries.

This is alas <u>not</u> a sufficient condition.

Ex: the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, $i^2=j^2=k^2=ijk=-1$ acts on $\mathbb{C}^2$ by

$$\pm 1 \mapsto \pm \mathrm{Id}, \quad \pm i \mapsto \pm\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}, \quad \pm j \mapsto \pm \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \quad \pm k \mapsto \pm\begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}$$

$\chi(\pm 1) = \pm 2$, all others have $\chi = 0$: so χ takes real values.

However this does not come from a 2-dimensional real representation: $Q \not\hookrightarrow GL(2,\mathbb{R})$.
(This is because a real representation of a finite group has an invariant inner product, so we'd get $Q \hookrightarrow O(2)$, with -1 acting by $-\mathrm{Id}$, but only 2 elements of $O(2)$ square to $-\mathrm{Id}$ (rotations by $\pm 90°$) while we need 6 such elements for $\pm i, \pm j, \pm k$.)

- To use characters etc. to classify rep^{ns}/$\mathbb{R}$, we need to understand which rep^s over $\mathbb{C}$ are real! We'll figure this out now for irreducible reps. over $\mathbb{C}$. However, beware: if V_0 is an irred. rep. over $\mathbb{R}$, then $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$ can still be reducible over $\mathbb{C}$. (Ex: $\mathbb{Z}/n$ rotations of $\mathbb{R}^2$)

Prop: A complex representation V is real iff there exists a G-equivariant, complex antilinear map $\tau: V \to V$ (ie. $\tau(\lambda v) = \bar{\lambda}\tau(v)$) such that $\tau^2 = id$.

Pf: One direction is clear: if $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$, let $\tau(v+iw) = v - iw$ for $v, w \in V_0$: complex conjugation! In opposite direction, given τ, $v \in V$ decomposes into $Re(v) = \frac{v+\tau(v)}{2}$ and $i\,Im(v) = \frac{v - \tau(v)}{2}$ which belong to the ± 1 eigenspaces of τ. Let $V_0 = \ker(\tau - id)$, which is an $\mathbb{R}$-subspace of V (not a $\mathbb{C}$-subspace!) and, as $\mathbb{R}$-linear maps, $\tau i = -i\tau$ so iV_0 is the -1-eigenspace, and $V = V_0 \oplus iV_0 \simeq V_0 \otimes_{\mathbb{R}} \mathbb{C}$.

The above was just linear algebra, but G-equivariance of τ implies that the eigenspace $V_0 = \ker(\tau - 1)$ is preserved by G, hence a subrep. over $\mathbb{R}$ (similarly for iV_0). $\square$

- Now, let V be an irreducible complex rep. of G, such that χ_V takes values in $\mathbb{R}$. Then $\chi_V = \overline{\chi_V} = \chi_{V^*}$, so $V \simeq V^*$ as G-reps.

Let $\varphi: V \xrightarrow{\sim} V^*$ such an iso. (by Schur φ is unique up to multiplication by some $\lambda \in \mathbb{C}^*$).

Recall: a linear map $\varphi: V \to V^*$ determines a bilinear form $B: V \times V \to \mathbb{C}$, $B(v,w) = \varphi(v)(w)$.

$B(gv, gw) = \varphi(gv)(gw)$ vs. $B(v,w) = (\varphi(v) \circ g^{-1})(gw) = \underbrace{(g\varphi)(v)(gw)}_{G\text{-action on } V^*}$ $\Rightarrow B$ is G-inv^t iff φ is equiv^t.

Hence: V admits a G-invariant bilinear form B, unique up to scaling, and nondeg. if nonzero. Now, recall $B \in (V \otimes V)^* = Sym^2 V^* \oplus \Lambda^2 V^*$, ie. the symmetric and skew parts of B $\left(= \frac{1}{2}(B(v,w) \pm B(w,v)\right)$ are also invariant. By uniqueness, one of these is zero and the other is nondegenerate; ie. B is either symmetric or skew.

The symmetric case corresponds to real rep^{ns}; the skew-symm. case gives quaternionic rep^s

Prop: An irreducible complex representation V of a finite group G is real iff V carries a G-invariant nondegenerate symmetric bilinear form $B: V\times V \to \mathbb{C}$.

Pf:
- Assume $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$ is real. Then V_0 has an invariant real inner product B; extend $\mathbb{C}$-bilinearly: $B(v_1+iw_1, v_2+iw_2) := B(v_1,v_2) + iB(w_1,v_2) + iB(v_1,w_1) - B(v_2,w_2)$ defines a nondegenerate symmetric bilinear form on V.
- Conversely: $B: V\times V \to \mathbb{C}$ determines an isom. $\varphi: V \to V^*$ ($\mathbb{C}$-linear, equivariant); choosing an invariant Hermitian inner product H on V, we also have a $\mathbb{C}$-antilinear equivariant bijection $V \to V^*$. Composing one with the inverse of the other gives a $\mathbb{C}$-antilinear equivariant map $\tau: V \to V$, characterized by: $H(\tau(v), w) = B(v,w)$.

  τ^2 is now an equivariant $\mathbb{C}$-linear isom. $V \to V$, hence $\tau^2 = \lambda\, Id$ by Schur.

  A calculation: $H(\tau^2(v), v) = B(\tau(v), v) = B(v, \tau(v)) = H(\tau(v), \tau(v)) \geq 0$ shows $\lambda \in \mathbb{R}_+$; replacing H by $\lambda^{1/2} H$ we can arrange $\tau^2 = id$.

  Thus V is real by the previous prop. $\square$

* In the other case where the invariant bilinear form B is skew-symmetric, the same argument gives a $\mathbb{C}$-antilinear equivariant bijective map $J: V \to V$ which now satisfies $J^2 = -id$. This is a <u>quaternionic</u> structure on V, ie. describes a structure of $\mathbb{H}$-module on V where $\mathbb{H}$ = quaternions $= \{a + bi + cj + dk \,/\, a,b,c,d \in \mathbb{R}\}$ $i^2 = j^2 = k^2 = ijk = -1$

"division algebra" (noncommutative analogue of a field: $\mathbb{H}$ is a noncommutative ring st. every nonzero element has a multiplicative inverse). $\mathbb{H} = \mathbb{C}1 \oplus \mathbb{C}j$, with $ji = -ij$, $j^2 = -1$, so an $\mathbb{H}$-module is the same thing as a $\mathbb{C}$-vector space + antilinear map j st. $j^2 = -id$.

Ex: the regular rep. V of S_3 is real. This can be seen directly if we notice that $S_3 \simeq D_3$ acts on $V_0 = \mathbb{R}^2$ by rotations and reflections, and $V_0 \otimes_{\mathbb{R}} \mathbb{C} \simeq V$...
or more abstractly by observing $V^* \simeq V$, and $\Lambda^2 V^* \simeq U'$ has no trivial summand hence $\nexists$ invariant skew-symmetric $B \in \Lambda^2 V^*$, but $\mathrm{Sym}^2 V^* \simeq U \oplus V$ has a trivial summand giving an invariant symmetric bilinear form $B \in \mathrm{Sym}^2 V^*$ & applying the above.

Ex: the 2-dim. representation of the quaternion group on $\mathbb{C}^2$ is quaternionic.
($\Leftrightarrow$ standard isom. $\mathbb{H} \simeq \mathbb{C} \oplus \mathbb{C}j \simeq \mathbb{C}^2$ with module structure $j(z_1 + z_2 j) = -\bar{z}_2 + \bar{z}_1 j$)