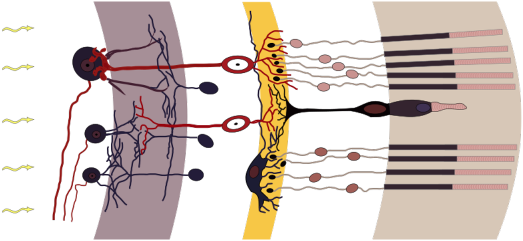
Seeing Single Photons

What a person can do

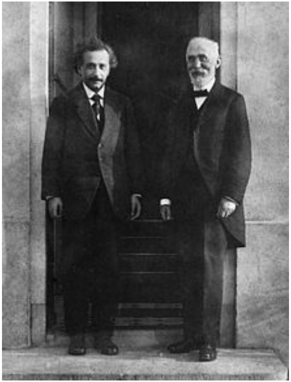
Prof. Aravi Samuel
David Zimmerman

Physics/Neuro 141

Week 2



Thresholds for seeing



Minimum energies for vision

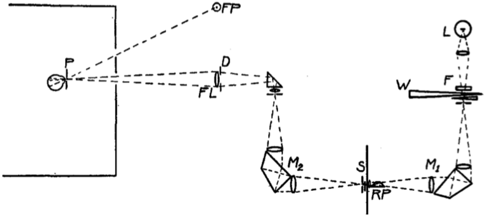
- $E_{min} = nhf$
- Lorentz calculated n 100 photons

Tapetum lucidum

Visual Conditions

- Dark adaptation
  - 30 minutes dark adaptation
- Peripheral vision
  - The greatest density of rods begins at 18°
- Small Test Fields
  - The larger the test field, the smaller the intensity that is needed
  - Reciprocal relation is not exact
  - Minimum product of area and intensity occurs at 10'
- Short exposures
  - The shorter the light pulse, the greater the intensity that is needed
  - Reciprocal relation is exact <0.01s
- The scotopic luminosity curve peaks at 510 nm

A set-up to test seeing



The eye at the pupil *P* fixates the red point *FP* and observes the test field formed by the lens *FL* and the diaphragm *D*. The light for this field comes from the lamp *L* through the neutral filter *F* and wedge *W*, through the double monochromator *MIM* and is controlled by the shutter *S*.

Observer	Energy	No. of quanta	Observer	Energy	No. of quanta
	ergs × 10[10]			*ergs × 10*[10]	
S. H.	4.83	126	C. D. H.	2.50	65
	5.18	135		2.92	76
	4.11	107		2.23	58
	3.34	87		2.23	58
	3.03	79			
	4.72	123	M. S.	3.31	81
	5.68	148		4.30	112
S. S.	3.03	79	S. R. F.	4.61	120
	2.07	54			
	2.15	56	A. F. B.	3.19	83
	2.38	62			
	3.69	96	M. H. P.	3.03	79
	3.80	99		3.19	83
	3.99	104		5.30	138

Each datum is the result of many measurements during a single experimental period,and is the energy which can be seen with 60 percent frequency. $\lambda = 510$nm, $E_{photon} = 3.84 \times 10^{-19}$J

.

Binomial and Poisson Statistics

The binomial distribution

- Flip a biased coin, with a probability of a head p and the probability of a tail $1-q$.
- The probability of a given sequence, e.g., 10010..., is $pqqpq\dots$
- There are a total of 2^n possible sequences. The number of these with k heads and $n-k$ tails is:

$$\frac{n!}{k!(n-k)!} = \binom{n}{k}$$

- Thus, the probability of exactly k heads in n flips, i.e., **the binomial distribution**, is

$$P(k;n,p) = \binom{n}{k} p^k q^{n-k}$$

- We can use the binomial theorem to show that the binomial distribution is normalized:

$$\sum_{k=0}^{n} = P(k;n,p) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p+q)^n = 1^n = 1$$

Expectation values

- The expectation value of k is its mean value:

$$\langle k \rangle = \sum_{k=0}^{n} kP(k;n,p) = \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k q^{n-k}$$

- To evaluate this, note that when $k = 0$, the term is 0, and that $k/k! = 1/(k-1)!$

$$\langle k \rangle = \sum_{k=0}^{n} \frac{n!}{(k-1)!(n-k)!} p^k q^{n-k}$$

- Factor out np:

$$\langle k \rangle = np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} q^{n-k}$$

- Change variables by substituting $m = k - 1$ and $s = n - 1$:

$$\begin{aligned} \langle k \rangle &= np \sum_{m=0}^{s} \frac{s!}{m!(s-m)!} p^m q^{s-m} \\ &= np \sum_{m=0}^{s} \binom{s}{m} p^m q^{s-m} = np = 1 \end{aligned}$$

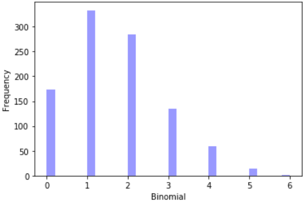
The binomial distribution

```
In [1]: import seaborn as sb, numpy as np

In [2]: from scipy.stats import binom

In [22]: data_binom = binom.rvs(n=10,p=0.16,loc=0,size=1000)
         ax = sb.distplot(data_binom,kde=False, color='blue')
         ax.set(xlabel='Binomial', ylabel='Frequency')

Out[22]: [Text(0, 0.5, 'Frequency'), Text(0.5, 0, 'Binomial')]
```



The binomial distribution (frequency of k successes) for $n = 10$, $p = 0.16$, $N = 1000$ trials.

$$\langle k \rangle = np \qquad \sigma^2 = \langle k^2 \rangle - \langle k \rangle^2 = npq$$

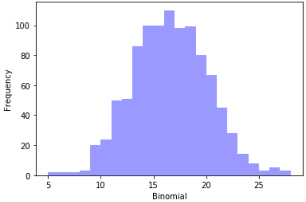
Binomial → Gaussian as n gets large

```
In [1]: import seaborn as sb, numpy as np

In [2]: from scipy.stats import binom

In [23]: data_binom = binom.rvs(n=100,p=0.16,loc=0,size=1000)
         ax = sb.distplot(data_binom,kde=False, color='blue')
         ax.set(xlabel='Binomial', ylabel='Frequency')

Out[23]: [Text(0, 0.5, 'Frequency'), Text(0.5, 0, 'Binomial')]
```



The binomial distribution (frequency of k successes) for $n = 100$, $p = 0.16$, $N = 1000$ trials *looks* like a Gaussian

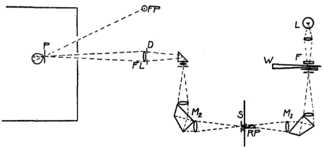
$$P(k; n, p) \to P(k; \mu, \sigma)dk = \frac{1}{\sqrt{2\pi\sigma^2}} e^{(-k-\mu)^2/2\sigma^2} dk$$

The Poisson distribution

- The Poisson distribution is obtained as an asymptotic limit of the binomial distribution when p is very small

$$P(k;\mu) = \frac{\mu^k}{k!} e^{-\mu}$$

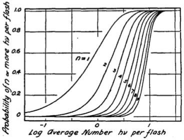
- One free variable: $\sigma = \sqrt{\mu}$
- Probability of zero successes: = $e^{-\mu}$
- Probability of more than one success: $1 - e^{-\mu}$



Large numbers of photons are released from the lamp, small numbers succeed in being “seen”

Poisson statistics in seeing

Probability of seeing



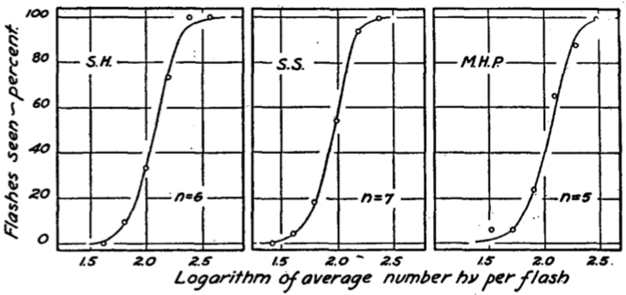
Poisson probability distributions. For any average number of quanta per flash, the ordinates give the probabilities that the flash will deliver to the retina or more quanta, depending on the value assumed for n.

- Let a be the average number of quanta which a flash yields to the retina. Poisson statistics states that:

$$P_n = \frac{a^n}{n!} e^{-a}$$

- If θ is the threshold for seeing, then the probability of seeing is a cumulative probability distribution:

$$P_{see} = \sum_{n=\theta}^{\infty} \frac{a^n}{n!} e^{-a}$$



Relation between the average energy content of a flash of light (in number of photons) and the frequency with which it is seen by three observers.

The threshold

5 to 8 photons on a 10 minute circular field of the retina (about 500 rods)