METRIC SPACES

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1. Metric spaces

The goal of topology is to study those properties of geometric objects that are invariant under continuous deformation. This may sound kind of vague, but we'll see how to make this more precise as we go on. The definition of a topological space is one of the great achievements of modern mathematics; it encodes exactly the information we need about a space in order to study those of its properties that are invariant under continuous deformation, and it does so with remarkable efficiency. But the definition seemingly comes out of nowhere, unless one first develops some intuition. Therefore, we will start with a much more intuitive definition, that of a *metric space*, and see how it leads us to the notion of topological space.

A classical way to apprehend the geometry of any object is to consider distances between points. Considering distances between points of a space (elements of a set), and the axioms that should be satisfied, one arrives at the following definition:

Definition 1.1. A *metric space* (X, d) is a set X, together with a function $d : X \times X \to \mathbb{R}^{\geq 0}$, called the *distance function*, satisfying the conditions

(1) for any two points $p, q \in X$, $d(p, q) = 0$ if and only if $p = q$;
(2) for any two points $p, q \in X$, $d(p, q) = d(q, p)$; and
(3) (the triangle inequality) for any three points $p, q, r \in X$,

$$d(p, r) \leq d(p, q) + d(q, r).$$

Example 1.2. $X = \mathbb{R}^n$, with the usual distance function

$$d\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) = \Big(\sum (y_i - x_i)^2\Big)^{1/2}$$

Similarly, if $X \subset \mathbb{R}^n$ is any subset, we can take the distance function $d : X \times X \to \mathbb{R}^{\geq 0}$ on X to be the restriction to $X \times X$ of the standard distance function d on $\mathbb{R}^n$. (More generally, if X is any metric space and $Y \subset X$ any subset, Y inherits the structure of metric space from X; this is called the *induced metric*.)

Note that as an alternative to the euclidean distance function of Example 1.2, we could also take

$$d_\infty\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) = \max\{|y_i - x_i|\}$$

or

$$d_1\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) = \sum |y_i - x_i|,$$

called the *square metric* and the *diamond metric* respectively.

Exercise 1.3. Verify that the functions e and f satisfy the conditions of Definition 1.1; that is, $(\mathbb{R}^n, d_\infty)$ and $(\mathbb{R}^n, d_1)$ are metric spaces.

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2. Continuity and limits in metric spaces

Let X and Y be metric spaces with distance functions d_X and d_Y, and let $f : X \to Y$ be any map. We say that f is *continuous* at a point $p \in X$ if points close to p are mapped to points close to $f(p)$; more precisely,

Definition 2.1. Given metric spaces (X, d_X) and (Y, d_Y), $f : X \to Y$ is continuous at a point $p \in X$ if

$$\forall \varepsilon > 0,\ \exists\, \delta > 0 \text{ s.t. } \forall q \in X,\ d_X(q, p) < \delta \implies d_Y(f(q), f(p)) < \epsilon.$$

We say that $f : X \to Y$ is continuous if it is continuous at every point of X.

Definition 2.2. Let X be a metric space with distance d. We say that an infinite sequence of points $p_1, p_2, p_3, \ldots \in X$ has *limit* $p \in X$ if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \forall n > N,\ d(p_n, p) < \varepsilon.$$

This expresses the informal idea that the points p_n get and stay arbitrarily close to p.

There is a related notion of a *Cauchy sequence* of points in a metric space: this is a sequence $p_1, p_2, p_3, \ldots \in X$ whose members get and stay arbitrarily close to each other. More precisely:

Definition 2.3. A sequence $p_1, p_2, p_3, \ldots \in X$ is a *Cauchy sequence* if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \forall m, n > N,\ d(p_m, p_n) < \varepsilon.$$

Exercise 2.4.

(1) Use the triangle inequality to show that a sequence $p_1, p_2, p_3, \ldots$ of points in a metric space X can have at most one limit.
(2) Show that any sequence with a limit is Cauchy.
(3) Show by example that the converse is not true: a Cauchy sequence in a metric space need not have a limit.
(4) We say a metric space is *complete* if every Cauchy sequence has a limit. Find an example of a complete metric space.

3. Open sets

We now begin the process of extracting just those properties of a metric space that are invariant under continuous deformation. The crucial notion is that of an open set:

Definition 3.1. Let X be a metric space with distance d. We say that a subset $U \subset X$ is *open* if U contains all points of X sufficiently close to any given point of U: precisely,

$$\forall p \in U\ \exists \epsilon > 0\ :\ \forall q \in X,\ d(q, p) < \epsilon \implies q \in U.$$

There is a more intuitive way to say this, using the notion of an open ball. For a point p in a metric space X and any positive real number r, we define the *open ball* of radius r around p to be the set

$$B_r(p) \ = \ \{q \in X : d(q, p) < r\}.$$

In these terms, we can say a subset $U \subset X$ is open if it contains an open ball around each of its points.

Definition 3.2. A subset $Z \subset X$ is *closed* if the complement $U = X \setminus Z$ is open.

Beware: a subset of X can be both open and closed, or (more commonly) neither open nor closed.

Exercise 3.3.

(1) Use the triangle inequality to show that an open ball is indeed open.
(2) Show that a finite intersection of open subsets in a metric space is open; and that an arbitrary union of open subsets is open.
(3) Deduce from part (2) that an arbitrary intersection of closed subsets in a metric space is closed, and a finite union of closed subsets is closed.
(4) Show by example that an arbitrary intersection of open subsets need not be open.

The point of all this is that we can characterize continuous maps between metric spaces without explicitly invoking the metric: we have the key

Theorem 3.4. *Let X and Y be metric spaces. If $f : X \to Y$ is any map, then f is continuous if and only if the preimage $f^{-1}(U)$ of every open set $U \subset Y$ is open in X.*

Proof. First, suppose that f is continuous. The definition says that for any $\epsilon > 0$ and any point $p \in X$, the preimage of the ball $B_\epsilon(f(p))$ contains a ball $B_\delta(p)$. To show that the preimages under f of open sets are open, suppose that $U \subset Y$ is open. For any point $p \in f^{-1}(U)$, U will contain a ball $B_\epsilon(f(p))$ for some $\epsilon > 0$, so $f^{-1}(U)$ contains an open ball around p; thus $f^{-1}(U)$ is open.

The other direction is similar: if f is not continuous at $p \in X$, by definition there is an $\epsilon > 0$ such that the preimage $f^{-1}(B_\epsilon(f(p)))$ does not contain any open ball around p; thus $f^{-1}(B_\epsilon(f(p)))$ is not open. $\square$

There is an analogous characterization of limits in terms of open sets:

Exercise 3.5. Let X be a metric space. Show that a sequence $p_1, p_2, p_3, \ldots \in X$ of points in X has limit $p \in X$ if and only if every open set $U \subset X$ containing p contains all but finitely many of the points p_n.

4. Topological spaces

The point of all this is that notions like continuity, limits, etc. depend only on which sets are open, and not on a particular metric. If we want to forget the metric and just remember which sets are open, we are finally led to the definition of topological space:

Definition 4.1. A *topological space* is a set X, with a collection $\mathcal{T}$ of subsets of X called *open sets*, such that

(1) The intersection of finitely many open sets is open;
(2) The union of an arbitrary collection of open sets is open; and

(3) $\emptyset$ and $X \in \mathcal{T}$

We can then define continuous maps between topological spaces as suggested by Theorem 3.4:

Definition 4.2. Let X and Y be topological spaces. A map $f : X \to Y$ is said to be *continuous* if the preimage of any open subset of Y is open in X.

Limits can likewise be defined along the lines suggested in Exercise 3.5:

Definition 4.3. Let X be a topological space, and $p_1, p_2, p_3, \ldots \in X$ a sequence of points of X. We say that a point $p \in X$ is a *limit* of the sequence if every open subset of X containing p contains all but finitely many of the points p_i.

Note that there are some aspects of a metric space that are not so readily ported over to the setting of topological spaces: it's hard to see, for example, what the analogue of a Cauchy sequence would be in a topological space. And some things that are true for metric spaces may not hold for topological spaces in general: for example, the statement that a sequence has at most one limit fails for topological spaces in general.

Note also that different metrics on a set X can induce the same topology:

Exercise 4.4. Show that the metrics d and d_∞ on $\mathbb{R}^n$ defined in the first section induce the same topology on $\mathbb{R}^n$: that is, a subset $U \subset \mathbb{R}^n$ is open in one metric if and only if it is open in the other.

An important point: the definition of a topological space does a beautiful job of capturing the essential topological structure of a metric space, while stripping away extraneous information like the actual metric. But at the same time, it lets in the door a lot of spaces that are not in any reasonable sense "geometric." The standard first example of this is the space X consisting of two points p and q, with open sets $\emptyset$, $\{p\}$ and $\{p, q\}$. Roughly speaking, this means that "one point is infinitely close to the other, but not vice versa," which is not something that would normally arise in a geometric setting. (And yet, one does encounter strange topological spaces such as this one, for example, in algebraic geometry.)

Accordingly, one of our goals will be to understand what sorts of conditions we can impose on a topological space to ensure that it behaves like the sort of spaces that arise in geometry, and to which we want to apply the methods of topology. Again, an example: the *Hausdorff* condition on a topological space X says that for any two distinct points $p, q \in X$ there are disjoint open sets U and V containing p and q. This is clearly true in any metric space – if $d(p, q) = r$, take $U = B_{r/2}(p)$ and $V = B_{r/2}(q)$ – but fails for arbitrary topological spaces (like the two-point space above).