Recall: • a <u>metric space</u> (X,d) = set with distance function $d: X\times X \to \mathbb{R}_{\geq 0}$ st.
1) $d(p,q)=0$ iff $p=q$, 2) $d(p,q)=d(q,p)$, 3) $d(p,r)\leq d(p,q)+d(q,r)$

• <u>open balls</u> $B_r(p)=\{x\in X \,/\, d(p,x)<r\}$. $U\subset X$ is <u>open</u> iff $\forall p\in U\ \exists r>0$ st. $B_r(p)\subset U$.

$f: X\to Y$ is <u>continuous</u> $\iff \forall p\in X\ \forall \varepsilon>0\ \exists \delta>0$ st $f(B_\delta(p))\subset B_\varepsilon(f(p))$
$\iff \forall U\subset Y$ open, $f^{-1}(U)\subset X$ is open
↪ this will be the def^n outside of the metric case.

Recall: • a sequence $p_n\to p$ in (X,d) if $\forall\varepsilon>0\ \exists N$ st. $n\geq N \Rightarrow d(p_n,p)<\varepsilon$.

• **Prop:** if $p_n\to p$, then every open subset $U\ni p$ contains p_n for all but finitely many n.
This will be the definition of limit outside the metric case.
(Pf: $U\ni p$, U open $\Rightarrow \exists\,\varepsilon>0$ st. $B_\varepsilon(p)\subset U$. So $\exists N$ st. $n\geq N \Rightarrow p_n\in B_\varepsilon(p)\subset U$).

---

* We will now reformulate / generalize all this in the context of <u>topological spaces</u>, ie. sets equipped with a <u>topology</u> which may or may not come from a metric.

Def: A <u>topological space</u> = a set X together with a collection $\mathcal{T}\subset\mathcal{P}(X)$, the <u>open sets</u> in X, such that
• $\emptyset\in\mathcal{T}$, $X\in\mathcal{T}$
• arbitrary unions of open sets are open
• finite intersections of open sets are open.

<u>Why bother?</u> One answer: many natural topologies do not come from a metric! Eg, in analysis:
• on the space of (bounded) functions $f: X\to\mathbb{R}$, uniform convergence topology ($f_n\to f$ iff $\sup_x |f_n(x)-f(x)|\to 0$) comes from a metric $\left(d(f,g)=\sup_x |f(x)-g(x)|\right)$
but pointwise convergence ($f_n\to f$ iff $\forall x\in X\ f_n(x)\to f(x)$) doesn't. ("product topology")
• C^∞ topology on smooth functions $\mathbb{R}\to\mathbb{R}$ doesn't come from a metric either.

And on the other hand, a metric contains extraneous information for topology
Eg. $(\mathbb{R}^n,d)$, $(\mathbb{R}^n,d_1)$, $(\mathbb{R}^n,d_\infty)$ have the same open sets $\Rightarrow$ same top.

Def. • $f: X\to Y$ is <u>continuous</u> if $\forall U\subset Y$, U open $\Rightarrow f^{-1}(U)\subset X$ is open.
• a sequence $\{p_n\}$ in X converges to a <u>limit</u> p ($p_n\to p$) if $\forall U\ni p$ open, $\exists N\in\mathbb{N}$ st. $n\geq N \Rightarrow p_n\in U$.

Ex: • (X,d) metric space $\Rightarrow \mathcal{T}=\{U\subset X \,/\, \forall p\in U\ \exists\varepsilon>0$ st. $B_\varepsilon(p)\subset U\}$ metric topology
• <u>discrete</u> topology: $\mathcal{T}=\mathcal{P}(X)$ (every subset is open and closed.)
(this is in fact a metric topology: set $d(x,y)=1\ \forall x\neq y$.) (eg. usual top. on $\mathbb{Z}\subset\mathbb{R}$).

These abstract def's imply basic facts about continuity, such as:

Prop:
- if $f: X \to Y$ continuous, $p_n \to p$ in $X \Rightarrow f(p_n) \to f(p)$ in Y
- $f: X \to Y$ and $g: Y \to Z$ continuous $\Rightarrow g \circ f: X \to Z$ continuous. Exercise: check this.

---

* Given two topologies $\mathcal{T}, \mathcal{T}'$ on X, if $\mathcal{T} \subset \mathcal{T}'$ we say $\mathcal{T}'$ is finer, $\mathcal{T}$ is coarser.
The finest topology on X is the discrete one (all points are isolated), while the coarsest is $\{\emptyset, X\}$ ("one big clump").

- The finer topology $\mathcal{T}'$ has more open sets; it's easier for functions $X \to Y$ to be continuous wrt $\mathcal{T}'$ than $\mathcal{T}$ (every function from a discrete set is continuous)
It's harder for sequences to converge in $\mathcal{T}'$ (eg. on a discrete set, convergent sequences must be constant after finitely many terms; while for $\mathcal{T} = \{\emptyset, X\}$ every sequence converges to every point of X, in particular limit isn't unique!).

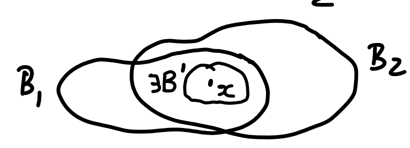
---

* Keeping track of all the open sets is cumbersome - in metric spaces we started with open balls & got a characterization of open sets in terms of these.
The analogous notion for a general topology is that of basis.

Def: Assume $\mathcal{B} \subset \mathcal{P}(X)$ is a collection of subsets of X st. 1) $\bigcup_{B \in \mathcal{B}} B = X$,

2) if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then $\exists\, B' \in \mathcal{B}$ st. $x \in B' \subset B_1 \cap B_2$.

Then we say $\mathcal{B}$ is a basis and generates the topology $\mathcal{T}$ = arbitrary unions of elements of $\mathcal{B}$.



Equivalently: $U \in \mathcal{T} \iff \forall x \in U\ \exists B \in \mathcal{B}$ st. $x \in B \subset U$.

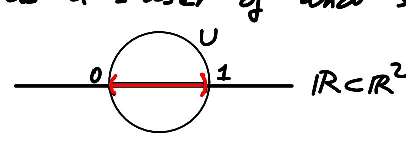
Check: (1) the two characterizations of $\mathcal{T}$ are equivalent, (2) $\mathcal{T}$ is a topology

Rmk: Unlike bases in lin. alg., bases in topology can contain redundant info - a better analogy is with generating sets... eg. metric topology is generated by any of:
all open sets; open balls $B_r(x)$, $x \in X$, $r > 0$; open balls $B_{1/n}(x)$, $x \in X$;
open balls $B_{1/n}(y)$, $y \in Y \subset X$ dense subset (every nonempty open intersects Y) eg. $\mathbb{Q} \subset \mathbb{R}$.
So for example the usual topology on $\mathbb{R}$ or $\mathbb{R}^n$ actually admits a countable basis!

- Making new topological spaces: subspaces, products.

Def: $(X, \mathcal{T}_X)$ top. space, $Y \subset X$ any subset $\Rightarrow$ the subspace topology on Y is
$\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}_X\}$. (Verify: this satisfies the axioms of a topology).

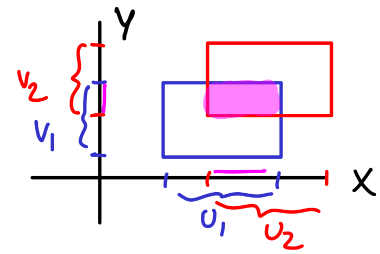
⚠ It's important when stating "U is open" to be clear: as a subset of what space?
Eg. Y is always open as a subset of itself!
$(0,1) \subset \mathbb{R} \subset \mathbb{R}^2$ is open in $\mathbb{R}$ but not in $\mathbb{R}^2$.



It's the coarsest topology on Y that makes the inclusion $Y \hookrightarrow X$ continuous. (HW).
Also, if T_X comes from a metric d on X, then T_Y comes from $d_{|Y} : Y \times Y \hookrightarrow X \times X \xrightarrow{d} \mathbb{R}_{\geq 0}$.

<u>Def. 2</u>: (X, T_X), (Y, T_Y) top. spaces $\Rightarrow$ the <u>product topology</u> on $X \times Y$ is the topology generated by basis $\mathcal{B} = \{ U \times V \ / \ U \subset X \text{ open}, \ V \subset Y \text{ open} \}$.

(Check $\mathcal{B}$ is a basis. Why isn't $\mathcal{B}$ already a topology?)



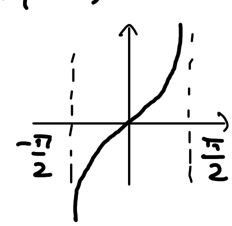
- When X, Y are metric spaces, this is also a metric topology, defined eg. by $d^{\infty}_{X \times Y}((x_1, y_1), (x_2, y_2)) = \max(d_X(x_1, x_2), d_Y(y_1, y_2))$
 (check! key observation: $B^{X \times Y}_r((x,y)) = B^X_r(x) \times B^Y_r(y)$).
 Or in fact $d^2_{X \times Y} = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$, $d^1_{X \times Y} = d_X(x_1, x_2) + d_Y(y_1, y_2)$ define the same topology on $X \times Y$ (see HW for case of $\mathbb{R}^2$). So this gives usual top. on $\mathbb{R}^n$.
- In general, it's the coarsest topology on $X \times Y$ st the projection maps $X \times Y \xrightarrow[p_1]{} X$, $X \times Y \xrightarrow[p_2]{} Y$ are continuous. (HW!)
 Also: $(x_n, y_n) \to (x,y)$ iff $x_n \to x$ and $y_n \to y$.
- Similarly for finite products $X_1 \times \dots \times X_n$. For infinite products there are several different natural topologies; see next week.

Homeomorphisms: what is the correct notion of 2 top. spaces being "the same"?

<u>Def</u>: X, Y are <u>homeomorphic</u> if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ st. $f \circ g = id_Y$, $g \circ f = id_X$.

Equivalently, a <u>homeomorphism</u> $f : X \to Y$ is a continuous bijection s.t. f^{-1} continuous.
Ie: a bijection $X \leftrightarrow Y$ under which $T_X \longleftrightarrow T_Y$.

<u>Rmk</u>:
- a continuous bijection need not be a homeomorphism
 <u>Eg</u>: X with 2 topologies, T' strictly finer than T $\Rightarrow$ $id_X : (X, T') \to (X, T)$ is a bijection, continuous since $U \in T \Rightarrow id^{-1}(U) = U \in T'$, but not homeo.
- say a metric space (X, d) is <u>bounded</u> if $\text{diam}(X) = \sup \{ d(p,q) \ / \ p, q \in X \} < \infty$
 This is <u>not</u> a top. property, eg. $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, $x \longmapsto \tan x$
 is a homeomorphism ($\tan$ & $\arctan$ are continuous),
 so $\mathbb{R}$ is homeo to $(\frac{-\pi}{2}, \frac{\pi}{2})$ (or any open interval in $\mathbb{R}$)



Closed sets: <u>Def</u>: a subset A of a topological space X is <u>closed</u> if $X \setminus A$ is open.

⚠ subsets can be both closed & open, eg. $\emptyset$ and X, or neither (eg. $[0,1)$ or $\mathbb{Q}$ in $\mathbb{R}$)

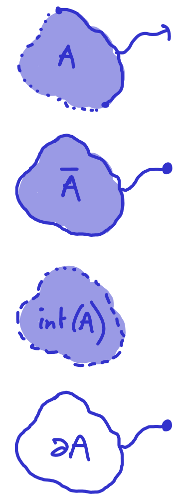
Axioms of open sets imply: $\begin{cases} \bullet\ \emptyset, X \text{ are closed} \\ \bullet \text{ arbitrary intersections of closed sets are closed} \\ \bullet \text{ finite unions of closed sets are closed.} \end{cases}$

<u>Def</u>: $A \subset X$ any subset $\Rightarrow$ we define

1) the <u>closure</u> of A: $\bar{A}$ = smallest closed set containing $A = \bigcap_{\substack{F \supset A \\ F \text{ closed}}} F$
   ($\bar{A} \supset A$, $\bar{A}$ closed since it's $\cap$ of closed)

2) the <u>interior</u> of A, $\operatorname{int}(A)$ = largest open set contained in A
   $= \bigcup_{U \subset A,\, U \text{ open}} U$ (open).

3) the <u>boundary</u> of A is $\partial A = \bar{A} - \operatorname{int}(A)$

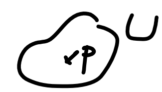


<u>Ex</u>: $A = [0,1) \subset \mathbb{R}$, usual top. $\Rightarrow$ $\bar{A} = [0,1]$, $\operatorname{int}(A) = (0,1)$, $\partial A = \{0,1\}$

<u>Rmk</u>: • A is closed iff $\bar{A} = A$, open iff $\operatorname{int}(A) = A$.

• $\overline{X \setminus A} = X \setminus \operatorname{int}(A)$, $\operatorname{int}(X - A) = X - \bar{A}$. $(*)$

<u>Def</u>: Say $U \subset X$ is a <u>neighborhood</u> of $p \in X$ if U is open and $p \in U$.



→ <u>Prop</u>: (1) $p \in \operatorname{int}(A)$ iff A contains a neighborhood of p.
(2) $p \in \bar{A}$ iff every neighborhood of p intersects A nontrivially.

(check this! (1) follows from def$^{\text{ns}}$: $p \in \operatorname{int}(A) \Leftrightarrow \exists U$ open s.t. $p \in U \subset A$.
(2) follows from (1) + $(*)$: $p \in \bar{A} \Leftrightarrow p \notin \operatorname{int}(X - A) \Leftrightarrow \forall U \ni p$ open, $A \cap U \neq \emptyset$).

<u>Def</u>: Say A is <u>dense</u> if $\bar{A} = X$. (ie. every nonempty open subset of X intersects A nontrivially).

<u>Ex</u>: $\mathbb{Q}$ is dense in $\mathbb{R}$ (for usual topology).

Next time we'll see the relation between closure, limit points, limits of sequences and introduce the Hausdorff property.