# Math 55b Homework 2

Due Wednesday February 10, 2021.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.
- Questions marked * may be on the harder side.

**Material covered:** Topologies on products; connectedness; compactness (Munkres §19-21, 23-27).

**0.** Sometime over the weekend of February 6-7, please complete the week 2 feedback survey (in Canvas). This is important to help us assess how well the course structure, pacing, and our efforts at getting students to know each other are working. (There will be more surveys).

**1.** The *Zariski topology* on $\mathbb{R}^2$ is the topology generated by the basis

$$U_f = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) \neq 0\}$$

where $f$ ranges over all polynomials in $\mathbb{R}[x, y]$.

(a) Show that the subsets $U_f \subset \mathbb{R}^2$ are indeed a basis for a topology.

(b) Show that this topology is not Hausdorff.

(c) What is the closure $\bar{I}$ of the line segment $I = \{(x, 0) \mid x \in [0, 1]\} \subset \mathbb{R}^2$ in the Zariski topology?

(d) If $\mathbb{R} \subset \mathbb{R}^2$ is the $x$-axis, show that the subspace topology on $\mathbb{R}$ induced by the Zariski topology is the finite complement topology.

**2.** Munkres exercise 19.6.

**3.** Munkres exercise 19.7.

**4.** Munkres exercise 23.9.

**5.** Let $\mathbb{R}_\ell$ denote the real line with the *lower limit topology*, generated by the basis consisting of all intervals $[a, b)$, $a < b$.

(a) Show that $\mathbb{R}_\ell$ is *totally disconnected*, i.e. its only (nonempty) connected subspaces are subsets consisting of a single point.

(b) Say a function $f : \mathbb{R} \to \mathbb{R}$ is *continuous from the right* (in the usual topology) if, $\forall a \in \mathbb{R}$, $\lim_{x \to a^+} f(x) = f(a)$, i.e. $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $a < x < a + \delta \Rightarrow |f(x) - f(a)| < \epsilon$. Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous from the right if and only if it is continuous when considered as a function from $\mathbb{R}_\ell$ to $\mathbb{R}$.

(c) What functions $f : \mathbb{R} \to \mathbb{R}$ are continuous when considered as maps from $\mathbb{R}$ to $\mathbb{R}_\ell$?

(d) What can you say about functions which are continuous as maps from $\mathbb{R}_\ell$ to $\mathbb{R}_\ell$?

**6.** Munkres exercise 24.1.

**7.** Munkres exercise 24.2.

**8.** Munkres exercise 24.3.

**9.** Munkres exercise 26.4.

**10.** Munkres exercise 26.5.

**11.** Let $X$ be the union of $\mathbb{R}^n$ and one additional point called $\infty$. Consider the topology with basis given by open balls in $\mathbb{R}^n$ plus the sets $U_r = \{\infty\} \cup \{x \in \mathbb{R}^n \mid |x| > r\}$. Show that $X$ is a compact Hausdorff space.

**12.** How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?