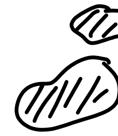
Def: A topological space X is <u>connected</u> if it cannot be written as $X = U \cup V$ where U, V are disjoint nonempty open sets. (such a decomposition is called a <u>separation</u> of X).



not connected.

Prop: $[0,1] \subset \mathbb{R}$ (standard top.) is connected. (proved last time)

Ex. $[0,1) \cup (1,2]$ is not connected, since $[0,1)$ and $(1,2]$ are open in subspace topology & provide a separation. More generally, $x<y<z \in \mathbb{R}$, $x,z \in A$, $y \notin A \Rightarrow A$ disconnected.

Thm: $f: X \to Y$ continuous, X connected $\Rightarrow f(X) \subset Y$ is connected.

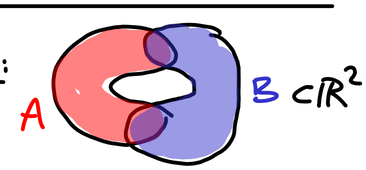
Pf: If $U \cup V$ is a separation of $f(X)$, then $f^{-1}(U) \cup f^{-1}(V)$ is a separation of X, contradiction! (subspace top.: $U = f(X) \cap U' \neq \emptyset$, U' open in $Y \Rightarrow f^{-1}(U) = f^{-1}(U') \neq \emptyset$ open in X; $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset$).

Corollary: <u>intermediate value theorem</u>

Theorem: X connected top space, $f: X \to \mathbb{R}$ continuous. If $a, b \in X$ and r lies between $f(a)$ and $f(b)$, then $\exists c \in X$ st. $f(c) = r$.

Pf. since X is connected, so is $f(X)$. If $r \notin f(X)$ then $U = (-\infty, r) \cap f(X)$ and $V = (r, \infty) \cap f(X)$ gives a separation of $f(X)$ (one contains $f(a)$ and the other contains $f(b)$) - contradiction. So $r \in f(X)$. $\square$

Fact: $A, B \subset X$ connected (for subspace top.) $\not\Rightarrow A \cap B$ connected. **Ex:**



But things are better for <u>unions</u> of connected sets, provided they overlap.

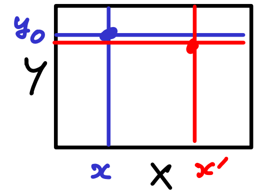
Thm: $A_i \subset X$ connected subspaces, all containing some point $p \in X$ (ie. $\cap A_i \neq \emptyset$). Then $Y = \cup A_i$ is connected.

Pf: assume $Y = U \cup V$ disjoint, open in Y. Without loss of generality, $p \in U$. Then $U \cap A_i$ and $V \cap A_i$ are disjoint, open in A_i. Since A_i is connected and $p \in U \cap A_i$, must have $A_i \subset U$ $\forall i$. Hence $Y = \cup A_i \subset U$ (and $V = \emptyset$). So Y is connected. $\square$

Corollary: $\mathbb{R}$ is connected; so are open, half-open, and closed intervals in $\mathbb{R}$.

Thm: X, Y connected $\Rightarrow X \times Y$ is connected.

Pf: Fix $(x_0,y_0)\in X\times Y$. Then $\forall x\in X$, $A_x=\left(X\times\{y_0\}\right)\cup\left(\{x\}\times Y\right)$ is connected by previous thm (both pieces contain (x,y_0)) and now $X\times Y=\bigcup_{x\in X}A_x$ (all containing (x_0,y_0)) $\Rightarrow X\times Y$ connected. $\square$

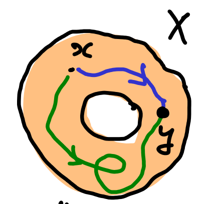


In fact, more is true: ‖ X_i, $i\in I$ connected $\Rightarrow \prod_{i\in I}X_i$ with product top is connected.
(won't prove).

(This is false for uniform and box topologies: eg $\mathbb{R}^I=\{\text{functions } I\to\mathbb{R}\}$ for infinite I. Say $f:I\to\mathbb{R}$ is bounded if $f(I)\subset\mathbb{R}$ bounded subset. Then $\{\text{bounded}\}\cup\{\text{unbounded}\}$ is a separation of $\mathbb{R}^I$ in uniform topology.).

Path-connectedness:

Def: ‖ X top. space, $x,y\in X$, a path from x to y is a continuous map $f:[a,b]\to X$ st. $f(a)=x$ and $f(b)=y$.
↑ subspace top. of standard $\mathbb{R}$



two paths $x\to y$.

Def: ‖ X is path-connected if every pair of points in X can be joined by a path.

Note: The relation $x\sim y \iff x$ and y can be connected by a path is an equivalence relation, ie.
(1) $x\sim x$ (constant path $f(t)=x$)
(2) $x\sim y \iff y\sim x$ (backwards path $f(-t)$)
(3) $x\sim y$ and $y\sim z \Rightarrow x\sim z$
(concatenate paths: $f=\begin{cases} f_1(t) & t\in[a,c]\\ f_2(t) & t\in[c,b]\end{cases}$)

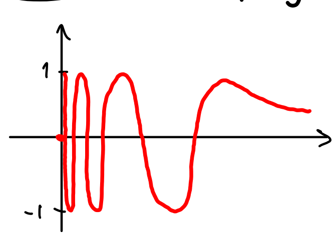
The equivalence classes are called the path components of X. (will return to these in alg. topology!)

Thm: ‖ if X is path connected then X is connected.

Pf: Assume not, ie $X=U\sqcup V$ disjoint open, $x\in U$, $y\in V$. Pick a path $f:[a,b]\to X$ connecting x to y. Then $[a,b]=\underset{\substack{\cup\!\!\shortmid\\ a}}{f^{-1}(U)}\sqcup \underset{\substack{\cup\!\!\shortmid\\ b}}{f^{-1}(V)}$ open subsets. This contradicts the connectedness of $[a,b]$. $\square$

The converse is false in general, but true for nice enough spaces eg. CW-complexes.

Ex: the "topologist's sine curve": let $S=\underbrace{\{(x,y)\mid y=\sin(\tfrac{1}{x}),\ x>0\}}_{\text{def}:=S_0}\cup\{(0,0)\}\subset\mathbb{R}^2$.



the "main" part S_0 is connected, since it's the image of $\mathbb{R}_+$ (connected) under the continuous map $x\mapsto(x,\sin\frac{1}{x})$.

Hence S is connected: if $S = U \cup V$ disjoint open, then

$S_0 = (U \cap S_0) \cup (V \cap S_0)$ disjoint & open $\Rightarrow$ one of them (eg. $V \cap S_0$) is empty.

$V \subset S - S_0 = \{(0,0)\}$. But $\{(0,0)\}$ not open in S, so in fact $V = \emptyset$.

On the other hand, S is not path connected: there's no path connecting $(\frac{1}{\pi}, 0)$ to $(0,0)$.
(Pf. later using compactness: the image of such a path is a closed subset of $\mathbb{R}^2$, but S isn't: $(0,1)$ is a limit point of S not in S).

However, for nice enough spaces the two notions are equivalent.

<u>Thm</u>: $A \subset \mathbb{R}^n$ open $\Rightarrow$ A is connected iff A is path connected.

<u>Pf</u>: already seen: path connected $\Rightarrow$ connected. We show: not path connected $\Rightarrow$ not connected.
Assume A open in $\mathbb{R}^n$: then the path components of A are open.
Indeed, if $x \in A$ then $\exists\, r > 0$ st. $B_r(x) \subset A$, and any two points of $B_r(x)$ can be connected inside A by a straight line segment. So all of $B_r(x)$ is in the same path component. Now: if A is not path connected then
$A = (\text{one path component}) \cup (\bigcup \text{all other path components})$ gives a separation. $\square$

This implies similar results for other classes of spaces, eg. top. manifolds and CW-complexes.

* For these kinds of space, path-components are also connected components, ie. they give a partition of X into disjoint connected open (& closed) subsets.
Such a partition only exists if X is "locally connected" ie. the topology has a basis consisting of connected open subsets. (Counterexample: $\mathbb{Q} \subset \mathbb{R}$ isn't loc. conn.)
(each point of $\mathbb{Q}$ is its own path component, but these aren't open).

<u>Compactness</u> (Munkres §26-...)

Compactness is a "finiteness/boundedness" property of nice topological spaces such as closed bounded intervals $[a,b] \subset \mathbb{R}$, or more generally, closed bounded subsets of $\mathbb{R}^n$.

<u>Eg</u>: any continuous map $f: \underset{\uparrow \text{compact}}{K} \to \mathbb{R}$ achieves its maximum & minimum.

The definition isn't very intuitive.

Def: An <u>open cover</u> of a top. space X is a collection of open subsets $(U_i)_{i \in I}$ st. $\bigcup_{i \in I} U_i = X$.

Def: X is compact if <u>every</u> open cover $(U_i)_{i\in I}$ of X admits a <u>finite subcover</u>, ie. $\exists\, i_1, \dots, i_n$ st. $X = U_{i_1} \cup \dots \cup U_{i_n}$.

Showing a space is <u>not</u> compact is much easier than showing it is!

<u>Ex:</u> $\mathbb{R}$ is not compact: the open cover $\mathbb{R} = \bigcup_{n\in\mathbb{N}} (n-1, n+1)$ has no finite subcover.

neither is $(0,1]$ with subspace top.: $(0,1] = \bigcup_{n\in\mathbb{N}} (\frac{1}{n}, 1]$ has no finite subcover.

<u>Ex:</u> $X = \{0\} \cup \{\frac{1}{n}, n \in \mathbb{Z}_+\}$ is compact: given <u>any</u> open cover $X = \bigcup_{i\in I} U_i$, let i_0 be such that $0 \in U_{i_0}$, then U_{i_0} also contains $\frac{1}{n}$ for all large $n \geq N$, hence $U_{i_1}, \dots, U_{i_N}$ containing $1, \frac{1}{2}, \dots, \frac{1}{N}$ and U_{i_0} give a finite subcover.

<u>Thm:</u> If X is compact and $f: X \to Y$ is continuous, then $f(X) \subset Y$ is compact
$\uparrow$ subspace top

(Rmk: an open cover of $f(X) \subset Y$ with subspace top. $\Leftrightarrow$ $U_i \subset Y$ open, $\bigcup_{i\in I} U_i \supset f(X)$).

<u>Pf:</u> let $\bigcup_{i\in I} U_i$ open cover of $f(X)$. Then $\bigcup_{i\in I} f^{-1}(U_i)$ is an open cover of X, hence $\exists i_1 \dots i_n$ st. $f^{-1}(U_{i_1}) \cup \dots \cup f^{-1}(U_{i_n}) = X$. So $\forall x \in X$ $f(x) \in U_{i_1} \cup \dots \cup U_{i_n}$, ie. $f(X) \subset U_{i_1} \cup \dots \cup U_{i_n}$ finite subcover. $\square$.

Once we know subsets of $\mathbb{R}^n$ are compact iff closed and bounded, taking $Y = \mathbb{R}$, this gives the extreme value theorem. To get started on this right away:

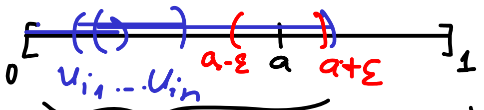
<u>Thm:</u> $[0,1]$ (with subspace top. $\subset \mathbb{R}$) is compact.

<u>Pf:</u> let $\{U_i\}_{i\in I}$ open cover of $[0,1]$.

Let $A = \{x \in [0,1] \mid \exists$ finite subcover $U_{i_1} \cup \dots \cup U_{i_n} \supset [0,x]\}$.

$A \neq \emptyset$ (contains 0). We want to show $1 \in A$. Let $a = \sup(A) \in [0,1]$.

- First we show $a \in A$: $\exists\, i_0$ st $a \in U_{i_0}$; since U_{i_0} is open, $\exists \varepsilon > 0$ st. $B_\varepsilon(a) \subset U_{i_0}$. On the other hand $a = \sup A$, so $\exists x \in A$ st. $x > a - \varepsilon$, and a finite subcover $[0,x] \subset U_{i_1} \cup \dots \cup U_{i_n}$. Therefore $[0,a] \subset U_{i_1} \cup \dots \cup U_{i_n} \cup U_{i_0}$, and $a \in A$.
- Next, assume $a < 1$: since $a \in A$, $\exists i_1 \dots i_n$ st. $[0,a] \subset U_{i_1} \cup \dots \cup U_{i_n}$, which is open, so $\exists \varepsilon > 0$ st. $B_\varepsilon(a) \subset U_{i_1} \cup \dots \cup U_{i_n}$, hence $U_{i_1} \cup \dots \cup U_{i_n}$ covers $[0, x]$ for some $x > a$ (eg. $x = a + \frac{\varepsilon}{2}$ if ≤ 1, else 1), contradicts $\sup(A) = a$.



finite cover of $[0, a+\varepsilon)$ (if $a + \varepsilon \leq 1$).

- So: $a = 1 \in A$, $\exists$ finite subcover. $\square$.

<u>Thm:</u> X compact, $F \subset X$ closed $\Rightarrow$ F is compact.

<u>Pf:</u> Given an open cover of F, ie. $U_i \subset X$ open, $\bigcup_{i\in I} U_i \supset F$, let $V = X \setminus F$ open: then $\{U_i, i \in I\} \cup \{V\}$ is an open cover of X, hence $\exists$ finite subcover. Discarding V, this gives a finite subcover for F. $\square$