Def: An *open cover* of a top. space X is a collection of open subsets $(U_i)_{i\in I}$ st. $\bigcup_{i\in I} U_i = X$.

Def: X is compact if *every* open cover $(U_i)_{i\in I}$ of X admits a *finite subcover*, ie. $\exists\, i_1,\dots,i_n$ st. $X = U_{i_1}\cup\dots\cup U_{i_n}$.

Ex: $\mathbb{R}$ is not compact: the open cover $\mathbb{R} = \bigcup_{n\in\mathbb{N}} (n-1,n+1)$ has no finite subcover.

Thm: $[0,1]$ (with subspace top. $\subset \mathbb{R}$) is compact.

(Proved last time). We'll now use this as starting point & eventually know enough to characterize compact subsets of $\mathbb{R}^n$.

Thm: X compact, $F\subset X$ closed $\Rightarrow$ F is compact.

Pf: Given an open cover of F, ie. $U_i\subset X$ open, $\bigcup_{i\in I} U_i \supset F$, let $V = X\setminus F$ open: then $\{U_i,\ i\in I\}\cup\{V\}$ is an open cover of X, hence $\exists$ finite subcover. Discarding V, this gives a finite subcover for F. $\square$

The converse is true in Hausdorff spaces!

Thm: X Hausdorff, $K\subset X$ compact $\Rightarrow$ K is closed in X.

Pf: We show that $X\setminus K$ is open. Let $x\in X\setminus K$.

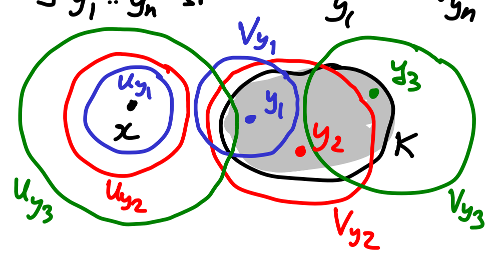
Since X is Hausdorff, $\forall y\in K\ \exists\, U_y\ni x,\ V_y\ni y$ disjoint open subsets.

Now $K\subset \bigcup_{y\in K} V_y$ is an open cover, so by compactness $\exists\, y_1,\dots,y_n$ st. $K\subset V_{y_1}\cup\dots\cup V_{y_n}$.

Let $U = U_{y_1}\cap\dots\cap U_{y_n}\ni x$ open.

Then $U\cap(V_{y_1}\cup\dots\cup V_{y_n}) = \phi$, so $U\cap K = \phi$.

Hence: $\forall x\in X\setminus K,\ \exists\, U$ open $\ni x$ st. $U\subset X\setminus K$. $\square$



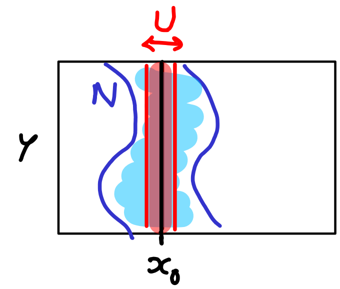
(If we tried this for an arbitrary subset of X, we'd find that $\bigcap_{y\in K} U_y$ isn't a neighborhood of x anymore. Compactness lets us reduce an infinite process to a finite one.)

Rmk: we've actually shown more: X Hausdorff, $K\subset X$ compact, $x\in X\setminus K \Rightarrow \exists$ disjoint open subsets $U\ni x$, $V\supset K$, $U\cap V=\phi$. Ie: can separate points from compact subsets!

Ex: When X isn't Hausdorff, $K\subset X$ compact $\not\Rightarrow$ K closed in X:

eg. $X=\mathbb{R}$ with finite complement top.; *any* subset $K\subset X$ is compact. Indeed, a nonempty open subset contains all but finitely many points, so given an open cover it is easy to find a finite subcover: take one nonempty U_i, with finite complement $\{p_1\cdots p_k\}$, then take U_{i_j} containing p_j for $j=1,\dots,k$.

Another instance of compactness allowing us to intersect infinitely many opens (or rather reduce to a finite intersection) is the tube lemma:



Prop: Let X top. space, Y compact top. space, $x_0 \in X$: if $N \subset X \times Y$ is open and $\{x_0\} \times Y \subset N$, then there exists a neighborhood U of x_0 in X st. $U \times Y \subset N$.

Pf: $\forall y \in Y$, $(x_0, y) \in N$ open, so $\exists$ basis open $U_y \times V_y$, U_y nbd. of x_0 in X, V_y nbd. of y in Y, st. $(x_0, y) \in U_y \times V_y \subset N$.

Now: $\bigcup_{y \in Y} V_y = Y$ open cover. (Rmk: $\left(\bigcap_{y\in Y} U_y\right) \times Y \subset N$, but $\bigcap_{y \in Y} U_y$ not open!)

Since Y is compact, $\exists y_1, \dots, y_n \in Y$ st. $Y = V_{y_1} \cup \dots \cup V_{y_n}$. Let $U = U_{y_1} \cap \dots \cap U_{y_n}$.

Then U is a neighborhood of x_0 in X, and $U \times Y \subset \bigcup_{i=1}^{n} U_{y_i} \times V_{y_i} \subset N$. $\square$

Thm: X, Y compact $\Rightarrow$ $X \times Y$ is compact.

Pf: Let $\{A_\alpha\}$ be an open cover of $X \times Y$. For any given $x \in X$, $\{x\} \times Y$ is compact so $\exists$ finite subcollection $A_{x,1}, \dots, A_{x,n(x)}$ which suffice to cover $\{x\} \times Y$.

$A_{x,1} \cup \dots \cup A_{x,n(x)}$ is open, so by the tube lemma $\exists U_x \ni x$ nbd. in X such that $A_{x,1} \cup \dots \cup A_{x,n(x)} \supset U_x \times Y$. Now X is compact, and $\{U_x\}_{x \in X}$ form an open cover, so $\exists x_1, \dots, x_k \in X$ st. $X = U_{x_1} \cup \dots \cup U_{x_k}$.

Now $A_{x_i, j}$ $1 \le i \le k$, $1 \le j \le n(x_i)$ is a finite subcover for $X \times Y$. $\square$

Theorem: $K \subset \mathbb{R}^n$ is compact iff K is closed and bounded.

Pf:
- if $K \subset \mathbb{R}^n$ is compact then it is closed (by above thm: $\mathbb{R}^n$ Hausdorff) and bounded: $K \subset \bigcup_{r > 0} B_r(0)$ open cover $\Rightarrow$ $\exists$ finite subcover $\Rightarrow$ $\exists R > 0$ st. $K \subset B_R(0)$.
- If $K \subset \mathbb{R}^n$ is closed and bounded, then it's a closed subset of $[-R, R]^n$ for some $R > 0$. $[-R, R]^n$ is a finite product of compact sets ($[-R,R] \simeq [0,1]$) hence compact; a closed subset of a compact is compact. $\square$

Rmk: • closed and bounded are necessary conditions for compactness of a subspace of any metric space (HW!) but in "most" metric spaces, closed + bounded $\not\Rightarrow$ compact.

There are easy counterexamples (find one for HW!). More interesting: let V be any infinite-dimensional vector space with a norm, $d(v, v') = \|v - v'\|$. Eg. $\mathcal{F} = C^0([a,b], \mathbb{R})$ continuous f$^{\text{ns}}$ with sup norm $d(f,g) = \sup |f - g|$. (uniform topology). Then $\bar{B} = \{v \in V \,/\, \|v\| \le 1\}$ is closed & bounded but never compact. (proof uses sequential compactness). (don't use this for Munkres 26.4)

We now look at applications of compactness. We've seen last time:

- **Thm:** If X is compact and $f: X \to Y$ is continuous, then $f(X) \subset Y$ (subspace top) is compact

Corollary: (extreme value theorem): X compact (nonempty), $f: X \to \mathbb{R}$ continuous $\Rightarrow$ f attains its max & min

Ex: (X,d) metric space, $A \subset X$ nonempty, $x \in X$ $\Rightarrow$ define $d(x,A) = \inf \{d(x,a) \mid a \in A\} \geq 0$ (distance of x to subset A).

If A is compact then the inf is always achieved! See HW3 Problem 1 = Munkres 27.2.

Similarly, the diameter of a bounded subset, $\operatorname{diam}(A) = \sup \{d(x,y) / x,y \in A\}$

The sup is attained for A compact ($d: A \times A \to \mathbb{R}$ continuous, achieves its max).

Another corollary: If X is compact and Y is Hausdorff, then any continuous bijection $f: X \to Y$ is a homeomorphism.

Pf: we need to check f^{-1} is continuous as well (so $U \subset X$ open $\Leftrightarrow$ $f(U) \subset Y$ open)

$U \subset X$ open $\Rightarrow$ $X \setminus U$ closed hence compact $\Rightarrow$ $f(X \setminus U) = Y \setminus f(U)$ compact

Since Y is Hausdorff this implies $Y \setminus f(U)$ is closed, ie. $f(U)$ open in Y. $\square$.

(We've seen that without such assumptions a continuous bijection need not be a homeo, eg. $[0,2\pi) \to S^1$, $t \mapsto (\cos t, \sin t)$).

In metric spaces, compactness implies uniform estimates.

- Lebesgue number lemma:

  (X,d) compact metric space, $(U_i)_{i \in I}$ open cover of X $\Rightarrow$ $\exists\, \delta > 0$ st. any subset of diameter $< \delta$ is entirely contained in a single open U_i.

Pf: by compactness, can assume (U_i) is a finite cover $= U_1 \cup \ldots \cup U_n$.

The function $f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, X \setminus U_i)$ is continuous (check: distance to a subset is a continuous function).

($X \setminus U_i$: closed set hence compact)

so achieves its min, which is therefore > 0 ($\forall x \in X$ $\exists i$ st. $x \in U_i$ and then $d(x, X - U_i) > 0$).

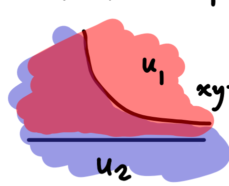
Hence $\exists\, \delta > 0$ st. $f(x) \geq \delta$ $\forall x \in X$. Thus $\forall x \in X$ $\exists\, U_i$ st. $d(x, X - U_i) \geq \delta$, ie. $B_\delta(x) \subset U_i$.

Since a subset of diameter $< \delta$ is contained in a ball of radius δ, the result follows. $\square$

This is the magic of compactness!

counterexamples: $\mathbb{R} = \bigcup$ intervals with overlaps of lengths $\to 0$ eg. $\bigcup_{n \in \mathbb{Z}} (n-1, n+1+\varepsilon_n)$, $\varepsilon_n \to 0$.

$\mathbb{R}^2 = U_1 \cup U_2$



distance $\to 0$.

This only makes sense for metric spaces! no notion of uniform size of neighborhood without a metric

Uniform continuity:

Def: $f: (X, d_X) \to (Y, d_Y)$ is uniformly continuous if $\forall \varepsilon > 0$, $\exists\, \delta > 0$ st. $\forall p, q \in X$, $d_X(p,q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon$.

(compare with continuity: the same δ must work for every p!).

Theorem: If X and Y are metric spaces, $f: X \to Y$ continuous, and X is compact, then f is uniformly continuous.

Proof: take $\varepsilon > 0$, and consider open cover of Y by balls of radius $\frac{\varepsilon}{2}$ (so if $f(p)$, $f(q)$ land in same ball, they're less than ε apart).

$X = \bigcup_{y \in Y} f^{-1}(B_{\varepsilon/2}(y))$ open cover, so by Lebesgue number lemma $\exists\, \delta > 0$ st. if $d_X(p,q) < \delta$ then they lie in the same element of the cover, hence $d_Y(f(p), f(q)) < \varepsilon$. $\square$

Alternative notions of compactness:

Def:
- X is limit point compact if every infinite subset of X has a limit point
- X is sequentially compact if every sequence $\{p_n\}$ in X has a convergent subsequence.

Ex: in $\mathbb{R}$, $\{\frac{1}{n},\ n \geq 1\} \cup \mathbb{Z}_+$ has a limit point (0) and the sequence $1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots$ has a convergent subsequence $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$

so does $0,1,0,1,0,1,\ldots$ (eg. subsequence $0,0,\ldots$).

but $\mathbb{Z} \subset \mathbb{R}$ has no limit point & the sequence $1,2,3,\ldots$ doesn't have a convergent subsequence, so $\mathbb{R}$ is neither limit point compact nor seq. compact.

Thm: X is compact $\Rightarrow$ X is limit point compact.

Pf: Assume X is not limit point compact, ie. $\exists\, A \subset X$ infinite with no limit point.
Since A contains all of its limit points (there are none), A is closed in X, hence compact.
However, $\forall a \in A$, a isn't a limit point so $\exists\, U_a \ni a$ neighborhood of a s.t. $U_a \cap A = \{a\}$.
$(U_a)_{a \in A}$ is now an infinite open cover of A, without any finite subcover since each $a \in A$ only belongs to U_a and not to any other element of the cover. Contradiction. $\square$

Thm: X sequentially compact $\Rightarrow$ X limit point compact.

Pf: Given $A \subset X$ infinite subset, pick a sequence of distinct points of A and take a convergent subsequence $\Rightarrow \exists\, \{a_n\}$ sequence in A, $a_n \neq a_m\ \forall n \neq m$, converging to some limit $a \in X$. Then every neighborhood of a contains a_n for all large n, hence ∞ly many points of A, including some $\neq a$. So a is a limit pt of A. $\square$

The converse implications don't hold in general, but in metric spaces all three notions coincide! (& hence also for subspaces of metric spaces...)

Thm: For a metric space (X,d), X compact $\Leftrightarrow$ X limit pt compact $\Leftrightarrow$ X seq. compact.