Compactness in metric spaces: gives access to uniform estimates (same $\delta>0$ works $\forall x\in X$). Eg:

- Lebesgue number lemma: (last time)

  Prop: (X,d) compact metric space, $(U_i)_{i\in I}$ open cover of $X \Rightarrow \exists\, \delta>0$ st. any subset of diameter $<\delta$ is entirely contained in a single open U_i.

- **Def:** $f:(X,d_X)\to(Y,d_Y)$ is uniformly continuous if $\forall \varepsilon>0,\ \exists\, \delta>0$ st. $\forall\, p,q\in X,\ d_X(p,q)<\delta \Rightarrow d_Y(f(p),f(q))<\varepsilon$.

  (compare with continuity: the same δ must work for every p!).

- **Theorem:** If X and Y are metric spaces, $f: X\to Y$ continuous, and X is compact, then f is uniformly continuous.

  Proof: take $\varepsilon>0$, and consider open cover of Y by balls of radius $\frac{\varepsilon}{2}$ (so if $f(p)$, $f(q)$ land in same ball, they're less than ε apart).

  $X=\bigcup_{y\in Y} f^{-1}(B_{\varepsilon/2}(y))$ open cover, so by Lebesgue number lemma $\exists\,\delta>0$ st. if $d_X(p,q)<\delta$ then they lie in the same element of the cover, hence $d_Y(f(p),f(q))<\varepsilon$. □

Alternative notions of compactness:

Def:
- X is compact if every open cover $(U_i)_{i\in I}$ of X has a finite subcover.
- X is limit point compact if every infinite subset of X has a limit point
- X is sequentially compact if every sequence $\{p_n\}$ in X has a convergent subsequence.

Ex: in $\mathbb{R}$, $\{\frac{1}{n},\ n\geq 1\}\cup\mathbb{Z}_+$ has a limit point (0) and the sequence $1,2,\frac{1}{2},3,\frac{1}{3},4,\frac{1}{4},\dots$ has a convergent subsequence $(\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots)$

so does $0,1,0,1,0,1,\dots$ (eg. subsequence $0,0,\dots$).

but $\mathbb{Z}\subset\mathbb{R}$ has no limit point & the sequence $1,2,3,\dots$ doesn't have a convergent subsequence, so $\mathbb{R}$ is neither limit point compact nor seq. compact.

Thm: X is compact $\Rightarrow$ X is limit point compact.

Pf. Assume X is not limit point compact, ie. $\exists\, A\subset X$ infinite with no limit point. Since A contains all of its limit points (there are none), A is closed in X, hence compact. However, $\forall a\in A$, a isn't a limit point so $\exists\, U_a\ni a$ neighborhood of a s.t. $U_a\cap A=\{a\}$. $(U_a)_{a\in A}$ is now an infinite open cover of A, without any finite subcover since each $a\in A$ only belongs to U_a and not to any other element of the cover. Contradiction. □

Thm: X sequentially compact $\Rightarrow$ X limit point compact.

Pf: Given $A \subset X$ infinite subset, pick a sequence of distinct points of A and take a convergent subsequence $\Rightarrow \exists \{a_n\}$ sequence in A, $a_n \neq a_m$ $\forall n \neq m$, converging to some limit $a \in X$. Then every neighborhood of a contains a_n for all large n, hence ∞ly many points of A, including some $\neq a$. So a is a limit pt of A. □

The converse implications don't hold in general, but in metric spaces all three notions coincide! (& hence also for subspaces of metric spaces...)

Thm: For a metric space (X,d), X compact $\Leftrightarrow$ X limit pt compact $\Leftrightarrow$ X seq. compact.

Proof: • compact $\Rightarrow$ limit point compact: already done (for all top spaces)

• limit point compact $\Rightarrow$ sequentially compact: suppose X metric space and limit point compact, and consider a sequence $x_1, x_2, \dots$ in X. If $\{x_1, x_2, \dots\}$ finite, then $\exists x \in X$ st. $x_n = x$ for infinitely many n, which gives a subsequence that converges to x.
Otherwise, $\{x_1, x_2, \dots\}$ is infinite, so has a limit point a. So:
$\forall r > 0$ $\exists n$ st. $0 < d(a, x_n) < r$.

First choose $n_1 \in \mathbb{N}$ st. $x_{n_1} \in B_1(a)$, then inductively, given $n_1, \dots, n_{k-1}$, let $\delta_k = \min\{d(x_i, a) \mid i \leqslant n_{k-1} \text{ and } x_i \neq a\} > 0$, and $r_k = \min(\frac{1}{k}, \delta_k)$.
Then take n_k st. $0 < d(a, x_{n_k}) < r_k$. By construction: $n_k > n_{k-1}$, and $d(a, x_{n_k}) < \frac{1}{k}$.
$\Rightarrow$ $x_{n_1}, x_{n_2}, \dots$ is a subsequence converging to a.

• seq. compact $\Rightarrow$ compact; this is the hardest part. First we show:

Lemma 1: If X metric space is seq. compact, then $\forall \varepsilon > 0$ X can be covered by finitely many open balls of radius ε.

(as we expect if X is to be compact: $X = \bigcup_{x \in X} B_\varepsilon(x)$ should have a finite subcover!)

Proof: assume not, and choose $x_1 \in X$, then inductively choose $x_n \in X \setminus \bigcup_{i=1}^{n-1} B_\varepsilon(x_i)$ (if this isn't possible then we've covered X by finitely many balls).
This yields a sequence in X, which by sequential compactness must have a convergent subsequence. But this is impossible since no two terms of the sequence are within ε of each other! Contradiction. □

Lemma 2: If X metric space is sequentially compact then every open cover has a Lebesgue number ($\exists \delta > 0$ st. any subset of diameter $< \delta$ is entirely in one U_i).

(we've seen this holds for compact metric spaces, so it should hold!)

Pf: suppose $\exists$ open cover $(U_i)_{i \in I}$ with no Lebesgue number, ie. $\forall n \geq 1$ $\exists C_n \subset X$ with diameter $< \frac{1}{n}$ which isn't contained in any single U_i. Take $x_n \in C_n$.

By sequential compactness, $\exists$ subsequence (x_{n_k}) of (x_n) that converges to some $a \in X$.

Now $a \in U_{i_0}$ for some $i_0 \in I$, and so $\exists\, \varepsilon > 0$ st. $B_\varepsilon(a) \subset U_{i_0}$

Take k suff'tly large so that $\frac{1}{n_k} < \frac{\varepsilon}{2}$ and $d(x_{n_k}, a) < \frac{\varepsilon}{2}$.

Since C_{n_k} has diameter $< \frac{\varepsilon}{2}$, $C_{n_k} \subset B_{\frac{\varepsilon}{2}}(x_{n_k}) \subset B_\varepsilon(a) \subset U_{i_0}$, contradiction. $\square$

(Rmk: this proof illustrates how arguments using sequential compactness are often more intuitive than those involving open covers: "if some property fails to hold uniformly, take a sequence of points where things get worse and worse, extract a convergent subsequence, and see what goes wrong at the limit.")

Now we can prove <u>seq. compact $\Rightarrow$ compact</u>:

<u>Pf</u>: Given an open cover $X = \bigcup_{i \in I} U_i$, by lemma 2 $\exists\, \delta > 0$ s.t. every subset of diameter $< \delta$ is entirely inside a single U_i. Fix $\varepsilon \in (0, \frac{\delta}{2})$: by lemma 1, X is covered by finitely many ε-balls. Each of these has diameter $\le 2\varepsilon < \delta$, so is contained in some U_i. This gives a finite subcover, replacing each ε-ball by one U_i containing it (and discarding the rest of the U_i's). $\square$.

<u>Thm</u>: Every compact metric space (X,d) is complete, ie. every Cauchy seq. converges.

<u>Pf</u>: let (x_n) Cauchy seq., by sequential compactness $\exists$ subsequence $x_{n_k} \to x \in X$.

Now $\forall \varepsilon > 0\ \exists N$ st. $\forall m, n \ge N$, $d(x_m, x_n) < \frac{\varepsilon}{2}$. $\exists\, n_k \ge N$ st. $d(x_{n_k}, x) < \frac{\varepsilon}{2}$.

Hence: $\forall n \ge N$, $d(x_n, x) \le d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon$. $\square$.

<u>Corollary</u>: $\mathbb{R}$, $\mathbb{R}^n$ (with usual distances) are complete.

<u>Pf</u>: every Cauchy sequence is bounded, hence contained in a compact subset, hence convergent. $\square$

<u>Corollary</u>: $\mathbb{R}^X = \{$ functions $X \to \mathbb{R}\}$ with uniform metric is complete.

<u>Pf</u>: given a Cauchy sequence $\{f_n\}$ (ie. $\forall \varepsilon > 0\ \exists N$ st. $m, n \ge N \Rightarrow \sup |f_n - f_m| < \varepsilon$).

$\forall x \in X$, $\{f_n(x)\}$ is a Cauchy seq. in $\mathbb{R}$ $\quad (|f_n(x) - f_m(x)| \le \sup |f_n - f_m| < \varepsilon)$

hence converges to some limit $f(x)$ (ie. we have a pointwise limit).

Now: given $\varepsilon > 0$, take N st. $m, n \ge N \Rightarrow \sup_x |f_n(x) - f_m(x)| < \varepsilon$.

Then $\forall n \ge N$, $\forall x \in X$, $|f_n(x) - f(x)| = \lim_{m \to \infty} |f_n(x) - f_m(x)| \le \varepsilon$.

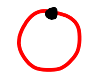
ie. $\forall n \ge N$, $\sup |f_n - f| \le \varepsilon$, which implies $f_n \to f$ uniformly. $\square$

(When X is a top. space, we've seen that uniform limits of continuous functions are continuous, so we also have completeness of $C^0(X, \mathbb{R}) = \{$continuous f$^{n}\} \subset \mathbb{R}^X$, uniform top.

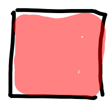
more generally: closed subsets of complete metric spaces are complete!)

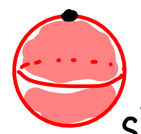
Compactification

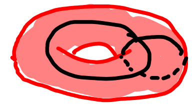
Def: A *compactification* of a (Hausdorff) top space X is a compact (Hausdorff) space Y with an inclusion $i: X \hookrightarrow Y$ which is an embedding (ie. homeom. onto its image, ie. topology on X = subspace topology of $i(X) \subset Y$), with X open & dense in Y ($\bar{X} = Y$).

Ex: $\mathbb{R}^n \rightsquigarrow \mathbb{R}^n \cup \{\infty\}$ as in HW2; this is in fact homeo to S^n (unit sphere in $\mathbb{R}^{n+1}$)
This is not the only option: eg. $(0,1) \simeq \mathbb{R}$ compactifies to  $[0,1]$ or  S^1

$(0,1) \times (0,1) \simeq \mathbb{R}^2$: eg.


$[0,1] \times [0,1]$


S^2


torus ($\simeq S^1 \times S^1$)

$\star$ The *one-point compactification*, if exists, is unique.

Let $Y = X \cup \{\infty\}$ (add a new point). The requirements of a compactification imply:
$\rightarrow$ a subset $U \subset X$ is open in Y iff it is open in X (subspace top. $= \mathcal{T}_X$)
$\rightarrow$ a subset V containing ∞ is open in Y iff $Y - V$ is closed, hence compact (we want Y compact), and a subset of X (since $\infty \in V$).

$\Rightarrow$ **Def:** $\mathcal{T}_Y = \{U \subset X \text{ open}\} \cup \{Y - K \,/\, K \subset X \text{ compact}\}$.

Thm: $\mathcal{T}_Y$ is a topology on $Y = X \cup \{\infty\}$, and Y is a compactification of X (in particular, Y is compact)

Pf: • axioms of a topology: case by case for U's and $(Y-K)$'s.
Arbitrary unions and finite $\cap$'s of a single type of open are still of the same type.
(note: $\bigcap (Y - K_i) = Y - (\bigcup K_i)$, a finite union of compact subsets of X is compact).
Moreover, $U \cap (Y - K) = U \cap (X - K)$ open $\subset X$
$U \cup (Y - K) = Y - (\underbrace{K \cap (X - U)})$ closed in K hence compact ✓

• Y is compact: if $(A_i)_{i \in I}$ open cover of Y, then $\infty \in A_{i_0} = Y - K$ for some $i_0 \in I$, and now the $(A_i \cap K)$ form an open cover of $K \Rightarrow \exists\, i_1 \dots i_n$ st. $A_{i_1} \cup \dots \cup A_{i_n} \supset K$.
Thus $Y = A_{i_0} \cup (A_{i_1} \cup \dots A_{i_n})$ finite subcover. □

However, this Y is not always Hausdorff! One-point compactif$^{\text{ns}}$ are only useful if Hausdorff.

Def: X is *locally compact* if $\forall x \in X$, $\exists K$ compact $\subset X$ which contains a neighborhood of x.

Ex: $\mathbb{R}$ is loc. compact ($x \in \mathbb{R} \Rightarrow x \in \operatorname{int}([x-1, x+1])$), so is $\mathbb{R}^n$.
$\mathbb{R}^\infty$ isn't (for any of usual topologies). Neither is $\mathbb{Q}$ with usual top ($\subset \mathbb{R}$)

Thm: The one-point compactif$^{\text{n}}$ $Y = X \cup \{\infty\}$ is Hausdorff iff X is locally compact and Hausdorff

Pf: • X Hausdorff $\Leftrightarrow$ we can separate points of $X \subset Y$ by open subsets (in X or in Y)
• X loc. compact $\Leftrightarrow$ $\forall x \in X$ $\exists$ opens $U \ni x$, $Y - K \ni \infty$ st. $U \subset K$ ie. $U \cap (Y - K) = \emptyset$
$\Leftrightarrow$ we can separate points of X from ∞ by open subsets in Y. □