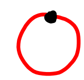
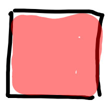
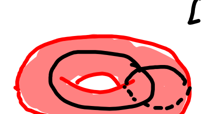
NOTE:
- midterm will be posted on Canvas Monday, due Friday Feb 19.
  topics = everything seen up to now (ending with compactifications)
  No collaboration / no materials other than lecture notes + Munkres.
- Monday is a holiday - no lecture / D.A.'s office hours may be cancelled.

Def: A <u>compactification</u> of a (Hausdorff) top space X is a compact (Hausdorff) space Y with an inclusion $i: X \hookrightarrow Y$ which is an embedding (ie. homeom. onto its image, ie. topology on X = subspace topology of $i(X) \subset Y$), with X open & dense in Y ($\bar{X} = Y$).

Ex: $\mathbb{R}^n \rightsquigarrow \mathbb{R}^n \cup \{\infty\}$ as in HW2; this is in fact homeo to S^n (unit sphere in $\mathbb{R}^{n+1}$)
This is not the only option: eg. $(0,1) \simeq \mathbb{R}$ compactifies to  $[0,1]$ or  S^1

$(0,1) \times (0,1) \simeq \mathbb{R}^2$: eg.  $[0,1] \times [0,1]$  S^2  torus ($\simeq S^1 \times S^1$)

★ The <u>one-point compactification</u>, if exists, is unique.
Let $Y = X \cup \{\infty\}$ (add a new point). The requirements of a compactification imply:
→ a subset $U \subset X$ is open in Y iff it is open in X (subspace top. $= \mathcal{T}_X$)
→ a subset V containing ∞ is open in Y iff $Y - V$ is closed, hence compact (we want Y compact), and a subset of X (since $\infty \in V$).

⇒ **Def:** $\mathcal{T}_Y = \{U \subset X \text{ open}\} \cup \{Y - K \,/\, K \subset X \text{ compact}\}$.

→ except: $\bar{X} = Y$ fails when X compact!

Thm: $\mathcal{T}_Y$ is a topology on $Y = X \cup \{\infty\}$, and Y is a compactification of X (in particular, Y is compact)

Pf:
- axioms of a topology: case by case for U's and $(Y-K)$'s.
  Arbitrary unions and finite $\cap$'s of a single type of open are still of the same type.
  (note: $\bigcap (Y - K_i) = Y - (\bigcup K_i)$, a finite union of compact subsets of X is compact).
  Moreover, $U \cap (Y - K) = U \cap (X - K)$ open $\subset X$
  $U \cup (Y - K) = Y - (K \cap (X - U))$ closed in K hence compact ✓
- Y is compact: if $(A_i)_{i \in I}$ open cover of Y, then $\infty \in A_{i_0} = Y - K$ for some $i_0 \in I$, and now the $(A_i \cap K)$ form an open cover of K ⇒ $\exists\, i_1 \dots i_n$ st. $A_{i_1} \cup \dots \cup A_{i_n} \supset K$.
  Thus $Y = A_{i_0} \cup (A_{i_1} \cup \dots A_{i_n})$ finite subcover. □

However, this Y is not always Hausdorff! One-point compactifᵒⁿˢ are only useful if Hausdorff.

Def: X is <u>locally compact</u> if $\forall x \in X$, $\exists K$ compact $\subset X$ which contains a neighborhood of x.

Ex: $\mathbb{R}$ is loc. compact ($x \in \mathbb{R} \Rightarrow x \in \mathrm{int}([x-1, x+1])$), so is $\mathbb{R}^n$.
$\mathbb{R}^\infty$ isn't (for any of usual topologies). Neither is $\mathbb{Q}$ with usual top ($\subset \mathbb{R}$)

Thm: The one-point compactif$^{\underline{n}}$ $Y = X \cup \{\infty\}$ is Hausdorff iff X is locally compact and Hausdorff.

Pf:
- X Hausdorff $\Leftrightarrow$ we can separate points of $X \subset Y$ by open subsets (in X or in Y)
- X loc. compact $\Leftrightarrow$ $\forall x \in X$ $\exists$ opens $U \ni x$, $Y-K \ni \infty$ st. $U \subset K$ ie. $U \cap (Y-K) = \emptyset$
  $\Leftrightarrow$ we can separate points of X from ∞ by open subsets in Y. $\square$

Countability axioms:

Def. X is first-countable if $\forall x \in X$, $\exists$ countable basis of neighborhoods at x, ie. $\exists U_1, U_2, \dots$ open $\ni x$ st. every neighborhood $V \ni x$ contains one of the U_n.

Ex: metric spaces are first-countable: at $x \in X$, take $U_n = B_{\frac{1}{n}}(x)$.

* In a first-countable space, $x \in \overline{A} \Leftrightarrow \exists$ sequence $x_n \in A$, $x_n \to x$. (else only $\Leftarrow$).

Def: X is second-countable if its topology has a countable basis.

Ex: $\mathbb{R}^n$ is second-countable, eg. basis: $\{B_r(x)\ \ x \in \mathbb{Q}^n,\ r \in \mathbb{Q}_+\}$ or $\{\prod (a_i, b_i) / a_i, b_i \in \mathbb{Q}\}$

$\mathbb{R}^\omega$ product top. is second-countable (basis = products of finite # of (a_i, b_i) $a_i < b_i \in \mathbb{Q}$ & all remaining factors are $\mathbb{R}$)

while uniform topology isn't (because $\exists$ uncountably many disjoint open subsets: balls of radius $1/2$ centered at $\{0,1\}^\omega$.)

* second-countable $\Rightarrow$ $\exists$ countable dense subset (eg: take one point in each basis open!)
the converse holds for metric spaces (take balls of radius $\frac{1}{n}$ around points of the dense subset)
but is false in general ($\mathbb{R}_\ell$ is first-countable, has countable dense subset, but $\nexists$ countable basis)

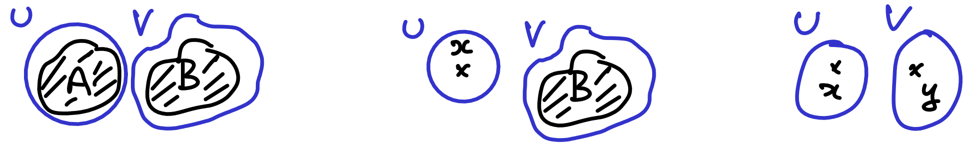
Regular and normal spaces (§31-32)

Recall: X Hausdorff := can separate points: $\forall x \neq y$, $\exists U \ni x, V \ni y$ disjoint open
(aka T_2) ($\Rightarrow T_1$: $\{x\}$ is closed $\forall x \in X$).

Stronger separation axioms:

Suppose one-point subsets $\{x\} \subset X$ are closed (T_1). Then say
- X is regular if $\forall x \in X$, $\forall B \subset X$ closed disjoint from x, $\exists$ disjoint open sets $U \ni x$, $V \supset B$.
- X is normal if $\forall A, B \subset X$ disjoint closed subsets, $\exists$ disjoint open sets $U \supset A$, $V \supset B$.

Metrizable $\Rightarrow$ Normal (T_4) $\Longrightarrow$ Regular (T_3) $\Longrightarrow$ Hausdorff (T_2) $\Longrightarrow$ T_1



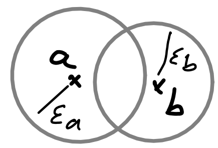
Ex: $\mathbb{R}_\ell$ is normal but not metrizable } see Munkres §31 for these and more.
$\mathbb{R}_\ell^2$ is regular but not normal.

Theorem:
- regular + second-countable $\Rightarrow$ normal
- Hausdorff + compact $\Rightarrow$ normal

(won't prove, cf. Munkres §32. We did see, when proving compact subsets of Hausdorff spaces are closed, that compact + Hausdorff $\Rightarrow$ regular; normal was an exercise on HW2. (Munkres 26.5)

Theorem: Metric spaces are normal.

Pf: Let A, B disjoint closed subsets $\subset (X,d)$.
$\forall a \in A$, $\exists\, \varepsilon_a > 0$ st. $B_{\varepsilon_a}(a) \subset X - B$.
$\forall b \in B$, $\exists\, \varepsilon_b > 0$ st. $B_{\varepsilon_b}(b) \subset X - A$.



Let $U = \bigcup_{a\in A} B_{\varepsilon_a/2}(a) \supset A$, $V = \bigcup_{b \in B} B_{\varepsilon_b/2}(b) \supset B$ (clearly open: $\cup$ balls)

We claim $U \cap V = \phi$. Indeed if $z \in U \cap V$ then $\exists\, a \in A$, $b \in B$ st.
$z \in B_{\varepsilon_a/2}(a) \cap B_{\varepsilon_b/2}(b)$. So $d(a,b) \leq d(a,z) + d(z,b) < \frac{\varepsilon_a}{2} + \frac{\varepsilon_b}{2} \leq \max(\varepsilon_a, \varepsilon_b)$.
But this is a contradiction (eg if $d(a,b) < \varepsilon_a$ then $B_{\varepsilon_a}(a) \not\subset X - B$!). $\square$

* We can now ask which topological spaces are metrizable.
We've seen: Metrizable $\Rightarrow$ first-countable and normal. ($\not\Leftarrow$ counterexample: $\mathbb{R}_\ell$)

Urysohn metrization theorem: If X is regular and has a countable basis, then it is metrizable.

(The first condition is necessary, the second one is stronger than needed. The Nagata-Smirnov thm gives a sharper criterion but is more technical to state & prove).

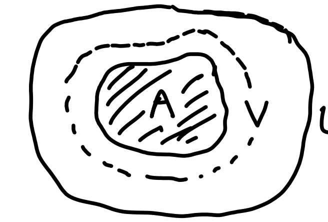
Urysohn's lemma is the key ingredient in the proof of the metrization theorem.

Thm: X normal space, A, B disjoint closed subsets
$\Rightarrow \exists$ continuous $f: X \to [0,1]$ st. $f(x) = 0\ \forall x \in A$ and $f(x) = 1\ \forall x \in B$.

Idea: 1) construct open sets U_q $\forall q \in [0,1] \cap \mathbb{Q}$ st. $A \subset U_0 \subset \dots \subset U_1 = X - B$
and moreover $p < q \Rightarrow \overline{U_p} \subset U_q$. & also set $U_q = X$ for $q > 1$.
2) define $f(x) = \inf\{q \in \mathbb{Q} \,/\, x \in U_q\}$. + show f is continuous.

Step 1 uses the following reformulation of normality:

Lemma: X is normal $\Rightarrow$ $\forall A$ closed, $\forall U \supset A$ open, $\exists$ open V st.
(in fact $\Leftrightarrow$) $A \subset V$ and $\overline{V} \subset U$.



Pf: A and $B = X - U$ are disjoint closed sets, so since X is normal, $\exists\, V \supset A$, $V' \supset B$ open such that $V \cap V' = \phi$.
Moreover, $X - V'$ closed, $V \subset X - V' \Rightarrow \overline{V} \subset X - V'$.
So $A \subset V \subset \overline{V} \subset X - V' \subset X - B = U$. □

Proof of Urysohn's lemma:

Step 1: Given A & B disjoint closed, let $U_1 = X - B$, and let U_0 open st. $A \subset U_0 \subset \overline{U}_0 \subset U_1$.

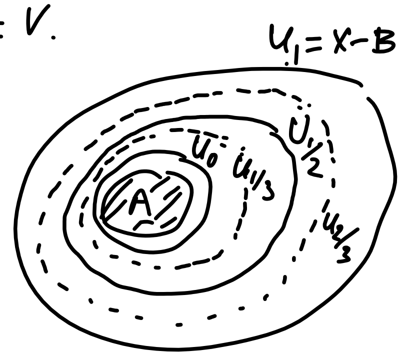
Next, we construct U_q, $q \in (0,1) \cap \mathbb{Q}$, st. $p < q \Rightarrow \overline{U}_p \subset U_q$ by induction:
choose a labelling of $[0,1] \cap \mathbb{Q} = \{q_0, q_1, q_2, q_3, \dots\}$ by an infinite sequence such that $q_0 = 0$ & $q_1 = 1$. (could eg. continue: $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \dots$).

Assuming $U_{q_0} \dots U_{q_n}$ have already been chosen, we construct $U_{q_{n+1}}$ using the above lemma:

let $q_k = \max\left(\{q_0 \dots q_n\} \cap [0, q_{n+1})\right)$ so $q_k < q_{n+1} < q_\ell$ & none of the
$q_\ell = \min\left(\{q_0 \dots q_n\} \cap (q_{n+1}, 1]\right)$ rationals already considered lie in between.

Then by induction hypothesis, $\overline{U_{q_k}} \subset U_{q_\ell}$, hence using normality
$\exists$ open V st. $\overline{U_{q_k}} \subset V \subset \overline{V} \subset U_{q_\ell}$, let $U_{q_{n+1}} = V$.

By induction, we construct in this way all the U_q's.
and indeed $p < q \Rightarrow \overline{U}_p \subset U_q$.



We also set $U_q = \phi$ if $q < 0$, X if $q > 1$.
(still true: $p < q \Rightarrow \overline{U}_p \subset U_q$!).

Step 2: Define $f(x) = \inf \mathbb{Q}_x$, where $\mathbb{Q}_x = \{q \in \mathbb{Q} \,/\, x \in U_q\}$.

Since $U_{<0} = \phi$ and $U_{>1} = X$, $(1, \infty) \subset \mathbb{Q}_x \subset [0, \infty)$ so $f(x) \in [0,1]\ \forall x \in X$

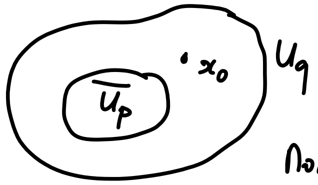
Also, $x \in A \subset U_0 \Rightarrow f(x) = 0$, and $x \in B \Rightarrow x \notin U_1 = X - B \Rightarrow \mathbb{Q}_x = (1, \infty)$ and $f(x) = 1$.

So: it only remains to show that $f: X \to [0,1]$ is continuous! For this, observe:

- $x \in \overline{U}_q \Rightarrow f(x) \leq q$: indeed if $x \in \overline{U}_q$ then $x \in U_{q'}\ \forall q' > q$ so $\mathbb{Q}_x \supset \mathbb{Q} \cap (q, \infty)$.
- $x \notin U_q \Rightarrow f(x) \geq q$: indeed if $x \notin U_q$ then $\mathbb{Q}_x \subset \mathbb{Q} \cap (q, \infty)$.

Now given an open interval (c,d), we show $f^{-1}((c,d))$ is open in X:

Assume $x_0 \in f^{-1}((c,d))$, and let $p, q \in \mathbb{Q}$ st. $c < p < f(x_0) < q < d$.



By the above observation, $x_0 \in U_q$ and $x_0 \notin \overline{U}_p$.

$V = U_q \cap (X - \overline{U}_p)$ is open, and a neighborhood of x_0.

Moreover, $x \in V \Rightarrow x \notin U_p$ so $f(x) \geq p$
$x \in \overline{U}_q$ so $f(x) \leq q$

Hence $V \subset f^{-1}([p,q]) \subset f^{-1}((c,d))$.
ie. $f^{-1}((c,d)) \supset$ nbds. of its points. □

Now we prove the metrization theorem, namely that if X is normal & has countable basis, then X is metrizable. We actually do this by embedding X as a subspace of a metric space, namely $[0,1]^\omega$ with product topology or uniform topology – in fact both come from metrics.

product top: $d((x_n),(y_n)) = \sup\{\frac{1}{n}|y_n - x_n|\}$ → then $B_\varepsilon((x_n)) = \prod_n (x_n - n\varepsilon, x_n + n\varepsilon)$

key point: for $n > \varepsilon^{-1}$ this is all of $[0,1]$.

<u>Step 1</u>: $\exists$ countable collection of continuous functions $f_n : X \to [0,1]$ st. $\forall x_0 \in X$, $\forall U \ni x_0$ neighborhood, $\exists n$ st. $f_n(x_0) > 0$ and $f_n \equiv 0$ on $X \setminus U$.

<u>Pf</u>: This follows from Urysohn's lemma, but need to be careful so that countably many functions suffice.

Let $\mathcal{B} = \{B_n\}$ countable basis for X. If $x_0 \in U$ open then $\exists B_n \in \mathcal{B}$ st. $x_0 \in B_n \subset U$.

But then, since X is normal, $\exists V$ open st. $x_0 \in V \subset \overline{V} \subset B_n$, and $\exists B_m \in \mathcal{B}$ st. $x_0 \in B_m \subset V$, so that $x_0 \in \overline{B}_m \subset B_n \subset U$.

So: for every $(m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ st. $\overline{B}_m \subset B_n$, apply Urysohn's lemma to get

$g_{m,n} : X \to [0,1]$ st. $g_{m,n} = 1$ on $\overline{B}_m$ and 0 on $X - B_n$.

This countable collection of functions has the stated property. □

<u>Step 2</u>: $F : X \to [0,1]^\omega$, product topology, $x \mapsto F(x) = (f_1(x), f_2(x), \dots)$ is an embedding, ie. continuous, injective, and X is homeo to $F(X) \subset [0,1]^\omega$

(so topology on X is defined by the metric $d_{|F(X)}$, QED)

<u>Pf</u>: • F is continuous in product topology because each component $f_1, f_2, \dots$ is continuous $X \to [0,1]$.

• F is injective, since $x \neq y \Rightarrow \exists U \ni x, V \ni y$ disjoint open

$\Rightarrow \exists m, n$ st. $f_n(x) > 0$, $f_n = 0$ outside of U (hence at y)

$f_m(y) > 0$, $f_m = 0$ outside of V (hence at x).

• finally, must show that F is a homeo $X \to Z = F(X) \subset [0,1]^\omega$. Since F is a continuous bijection $X \to Z$, only remains to prove: $U \subset X$ open $\Rightarrow F(U) \subset Z$ is open.

For this, let $U \subset X$ be any open set, and $x_0 \in U$. Then $\exists n$ st. $f_n(x_0) > 0$ and $f_n = 0$ outside of U. Let $V_n = \pi_n^{-1}((0,\infty)) \cap Z = \{z = (z_1, z_2, \dots) \in Z \,/\, z_n > 0\} \subset Z$ open

Then $x_0 \in F^{-1}(V_n) \subset U$ (since $f_n(x_0) > 0$, and $f_n(x) > 0 \Rightarrow x \in U$).

hence $F(x_0) \in V_n \subset F(U)$. This is true $\forall x_0 \in U$ ($\Leftrightarrow \forall F(x_0) \in F(U)$)

V_n open in Z. So we conclude that $F(U)$ is open.

Hence $F : X \to Z$ is a homeomorphism, and X is homeo to a metric space! □