Urysohn metrization theorem: If X is regular and has a countable basis, then it is metrizable.

(Recall: regular = can separate points from closed sets. We've stated that the assumptions imply X is normal ie. $\forall A, B \subset X$ disjoint closed subsets, $\exists$ disjoint open sets $U \supset A$, $V \supset B$.

Urysohn's lemma is the key ingredient in the proof of the metrization theorem.

Thm: X normal space, A, B disjoint closed subsets
$\Rightarrow \exists$ continuous $f: X \to [0,1]$ st. $f(x) = 0$ $\forall x \in A$ and $f(x) = 1$ $\forall x \in B$.

Idea: 1) construct open sets U_q $\forall q \in [0,1] \cap \mathbb{Q}$ st. $A \subset U_0 \subset \dots \subset U_1 = X - B$ & moreover $p < q \Rightarrow \overline{U_p} \subset U_q$.

2) define $f(x) = \inf \{q \in \mathbb{Q} / x \in U_q\}$. + show f is continuous.

Step 1 uses the following reformulation of normality:

Lemma: X is normal $\Rightarrow \forall A$ closed, $\forall U \supset A$ open, $\exists$ open V st. $A \subset V$ and $\overline{V} \subset U$.
(in fact $\Leftrightarrow$)

Pf: A and $B = X - U$ are disjoint closed sets, so since X is normal, $\exists V \supset A$, $V' \supset B$ open such that $V \cap V' = \emptyset$.
Moreover, $X - V'$ closed, $V \subset X - V' \Rightarrow \overline{V} \subset X - V'$.
So $A \subset V \subset \overline{V} \subset X - V' \subset X - B = U$. $\square$

Proof of Urysohn's lemma:

Step 1: Given A & B disjoint closed, let $U_1 = X - B$, and let U_0 open st. $A \subset U_0 \subset \overline{U_0} \subset U_1$.

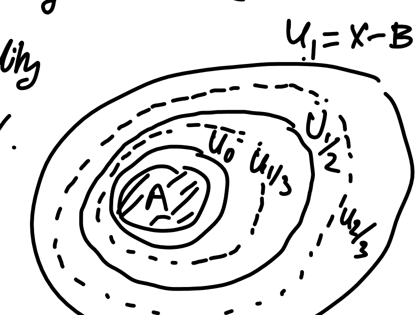
Next, we construct U_q, $q \in (0,1) \cap \mathbb{Q}$, st. $p < q \Rightarrow \overline{U_p} \subset U_q$ by induction:

choose a labelling of $[0,1] \cap \mathbb{Q} = \{q_0, q_1, q_2, q_3, \dots\}$ by an infinite sequence such that $q_0 = 0$ & $q_1 = 1$. (could eg. continue: $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \dots$).

Assuming $U_{q_0} \dots U_{q_n}$ have already been chosen, we construct $U_{q_{n+1}}$ using the above lemma:

let $q_k = \max(\{q_0 \dots q_n\} \cap [0, q_{n+1}))$ so $q_k < q_{n+1} < q_\ell$ & none of the
$q_\ell = \min(\{q_0 \dots q_n\} \cap (q_{n+1}, 1])$ rationals already considered lie in between.

Then by induction hypothesis, $\overline{U_{q_k}} \subset U_{q_\ell}$, hence using normality $\exists$ open V st. $\overline{U_{q_k}} \subset V \subset \overline{V} \subset U_{q_\ell}$, let $U_{q_{n+1}} = V$.



By induction, we construct in this way all the U_q's.
and indeed $p < q \Rightarrow \overline{U_p} \subset U_q$.

also set $U_q = \emptyset$ if $q < 0$, X if $q > 1$. (still true: $p < q \Rightarrow \overline{U_p} \subset U_q$!).

Step 2: Define $f(x) = \inf Q_x$, where $Q_x = \{ q \in \mathbb{Q} \,/\, x \in U_q \}$.

Since $U_{<0} = \phi$ and $U_{>1} = X$, $(1,\infty) \subset Q_x \subset [0,\infty)$ so $f(x) \in [0,1]$ $\forall x \in X$

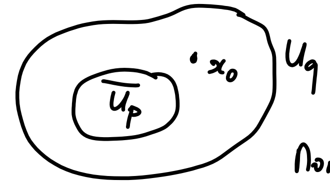
Also, $x \in A \subset U_0 \Rightarrow f(x) = 0$, and $x \in B \Rightarrow x \notin U_1 = X - B \Rightarrow Q_x = (1,\infty)$ and $f(x) = 1$.

So: it only remains to show that $f: X \to [0,1]$ is continuous! For this, observe:

- $x \in \overline{U_q} \Rightarrow f(x) \leq q$: indeed if $x \in \overline{U_q}$ then $x \in U_{q'}$ $\forall q' > q$ so $Q_x \supset \mathbb{Q} \cap (q,\infty)$.
- $x \notin U_q \Rightarrow f(x) \geq q$: indeed if $x \notin U_q$ then $Q_x \subset \mathbb{Q} \cap (q,\infty)$.

Now given an open interval (c,d), we show $f^{-1}((c,d))$ is open in X:

Assume $x_0 \in f^{-1}((c,d))$, and let $p, q \in \mathbb{Q}$ st. $c < p < f(x_0) < q < d$.



By the above observation, $x_0 \in U_q$ and $x_0 \notin \overline{U_p}$.

$V = U_q \cap (X - \overline{U_p})$ is open, and a neighborhood of x_0.

Moreover, $x \in V \Rightarrow x \notin U_p$ so $f(x) \geq p$; $x \in \overline{U_q}$ so $f(x) \leq q$. Hence $V \subset f^{-1}([p,q]) \subset f^{-1}((c,d))$.

ie. $f^{-1}((c,d)) \supset$ nbds. of its points. $\square$

Now we prove the metrization theorem, namely that if X is normal & has countable basis, then X is metrizable. We actually do this by embedding X as a subspace of a metric space, namely $[0,1]^\omega$ with product topology or uniform topology – in fact both come from metrics.

product top: $d((x_n),(y_n)) = \sup \{ \frac{1}{n} |y_n - x_n| \} \to$ then $B_\varepsilon((x_n)) = \prod_n (x_n - n\varepsilon, x_n + n\varepsilon)$

key point: for $n > \varepsilon^{-1}$ this is all of $[0,1]$.

Step 1: $\exists$ countable collection of continuous functions $f_n: X \to [0,1]$ st. $\forall x_0 \in X$, $\forall U \ni x_0$ neighborhood, $\exists n$ st. $f_n(x_0) > 0$ and $f_n \equiv 0$ on $X - U$.

Pf: This follows from Urysohn's lemma, but need to be careful so that countably many functions suffice.

Let $\mathcal{B} = \{B_n\}$ countable basis for X. If $x_0 \in U$ open then $\exists B_n \in \mathcal{B}$ st. $x_0 \in B_n \subset U$.

But then, since X is normal, $\exists V$ open st. $x_0 \in V \subset \overline{V} \subset B_n$, and $\exists B_m \in \mathcal{B}$ st. $x_0 \in B_m \subset V$, so that $x_0 \in \overline{B_m} \subset B_n \subset U$.

So: for every $(m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ st. $\overline{B_m} \subset B_n$, apply Urysohn's lemma to get $g_{m,n}: X \to [0,1]$ st. $g_{m,n} = 1$ on $\overline{B_m}$ and 0 on $X - B_n$.

This countable collection of functions has the stated property. $\square$

Step 2: $F: X \to [0,1]^\omega$, product topology, $x \mapsto F(x) = (f_1(x), f_2(x), \ldots)$ is an embedding, ie. continuous, injective, and X is homeo to $F(X) \subset [0,1]^\omega$

(so topology on X is defined by the metric $d_{|F(X)}$, QED)

<u>Pf:</u> • F is continuous in product topology because each component $f_1, f_2, \dots$ is continuous $X \to [0,1]$.

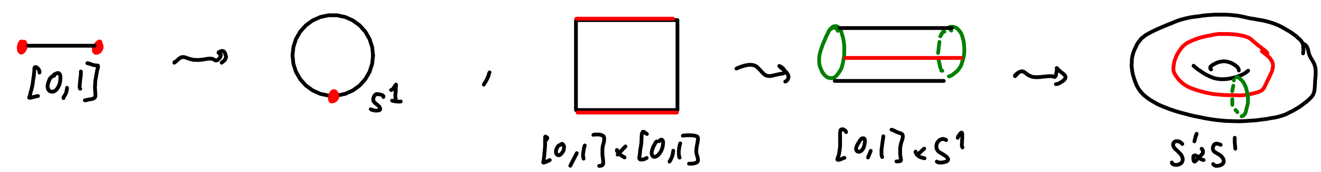
• F is injective, since $x \neq y \Rightarrow \exists\, U \ni x,\ V \ni y$ disjoint open
$\Rightarrow \exists\, m, n$ st. $f_n(x) > 0$, $f_n = 0$ outside of U (hence at y)
$f_m(y) > 0$, $f_m = 0$ outside of V (hence at x).

• finally, must show that F is a homeo $X \to Z = F(X) \subset [0,1]^\omega$. Since F is a continuous bijection $X \to Z$, only remains to prove: $U \subset X$ open $\Rightarrow$ $F(U) \subset Z$ is open.
For this, let $U \subset X$ be any open set, and $x_0 \in U$. Then $\exists\, n$ st. $f_n(x_0) > 0$ and $f_n = 0$ outside of U. Let $V_n = \pi_n^{-1}((0,\infty)) \cap Z = \{z = (z_1, z_2, \dots) \in Z \,/\, z_n > 0\} \subset Z$ open
Then $x_0 \in F^{-1}(V_n) \subset U$ (since $f_n(x_0) > 0$, and $f_n(x) > 0 \Rightarrow x \in U$).
hence $F(x_0) \in V_n \subset F(U)$. This is true $\forall x_0 \in U$ ($\Leftrightarrow \forall F(x_0) \in F(U)$)
V_n open in Z so we conclude that $F(U)$ is open.
Hence $F: X \to Z$ is a homeomorphism, and X is homeo to a metric space! □

<u>Gluing & quotients</u> (§22)

One good way to build interesting topological spaces is by "gluing" together simpler spaces.

<u>Ex:</u>



The construction underlying this is the <u>quotient topology</u>.

<u>Def:</u> X top. space, A a set, $f: X \to A$ a surjective map.
The <u>quotient topology</u> on A is defined by:
$U \subset A$ is open $\Leftrightarrow$ $f^{-1}(U) \subset X$ is open.

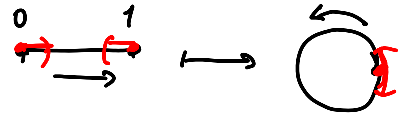
(<u>Exercise</u>: check this is a topology on A, in fact the finest topology for which f is continuous)

• A map $f: X \to Y$ between topological spaces is called a <u>quotient map</u> if f is surjective and $U \subset Y$ is open $\Leftrightarrow$ $f^{-1}(U) \subset X$ is open.
(ie. the topology on Y is the quotient topology induced by $f: X \to Y$)

• Typically, start from an equivalence relation $\sim$ on X, define A to be the set of equivalence classes $A = X/\sim$, and define $f: X \to X/\sim = A$, $x \mapsto [x]$.
Conversely, given any surjective map $f: X \to A$, we can define an equivalence relation on X by $x \sim x' \Leftrightarrow f(x) = f(x')$ and then $X/\sim = A$.

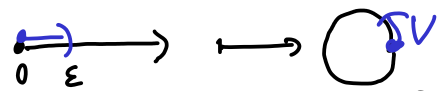
Ex: $S^1 \simeq [0,1]$ with 0 glued to 1: set $0 \sim 1$ so $\{0,1\}$ is one equiv. class.
(all others are just $\{x\}$).

The quotient map is $f: [0,1] \to S^1$
$t \mapsto (\cos 2\pi t, \sin 2\pi t)$



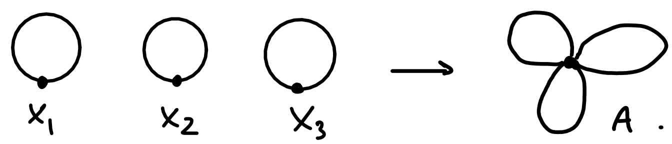
(Check! away from the end points f is a homeo $(0,1) \xrightarrow{\sim} S^1 - \{(1,0)\}$. so only need to check at 0 & 1. The point is: $U \ni (1,0)$ open in $S^1 \Leftrightarrow f^{-1}(U) \supset \{0,1\}$ open in $[0,1]$.)

vs. $g = f_{|[0,1)}: [0,1) \to S^1$ not a quotient map!



$V = g([0,\varepsilon))$ not open in S^1 vs. $g^{-1}(V) = [0,\varepsilon)$ open in $[0,1)$, want iff!
(whereas $f^{-1}(V) = [0,\varepsilon) \cup \{1\}$ not open in $[0,1]$ ✓)

Ex: $X_1, \dots, X_n$ top. spaces each $X_i \simeq S^1$, pick one point $x_i \in X_i$ $\forall i$.
+ let A = quotient space of $\coprod X_i$ by the equivalence relation $x_i \sim x_j$ $\forall i,j$.
(glue the X_i at their base points).. This is called the wedge of the circles $X_1, \dots X_n$.



★ There is a useful characterization of continuous maps from a quotient space:
if $A = X/\sim$ and $f: X \to Y$ is a map s.t. $x \sim x' \Rightarrow f(x) = f(x')$,
then we can define $\bar f: X/\sim \to Y$ by $\bar f([x]) = f(x)$.

$$\begin{tikzcd} & X/\sim \arrow[dr, "\bar f"] & \\ X \arrow[ur, "p"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Thm: If $f: X \to Y$ is a continuous map and $x \sim x' \Rightarrow f(x) = f(x')$, then equipping $X/\sim$ with the quotient topology, $\bar f: X/\sim \to Y$ is a continuous map.

Pf: let $p: X \to X/\sim$ the quotient map, and recall $\bar f([x]) = f(x)$
$x \mapsto [x]$ (indep. of $x \in [x]$).

So $\bar f \circ p = f$. Hence: $\forall U \subset Y$ open, $f^{-1}(U) = p^{-1}(\bar f^{-1}(U)) \subset X$ is open.

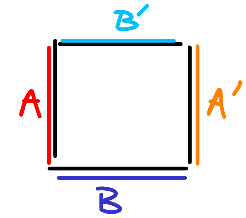
By definition of the quotient topology, we conclude that $\bar f^{-1}(U) \subset X/\sim$ is open. □
($V \subset X/\sim$ open $\Leftrightarrow p^{-1}(V) \subset X$ open).

(& conversely, since $p: X \to X/\sim$ is continuous. So: $\bar f$ continuous $\Leftrightarrow f = \bar f \circ p$ continuous)

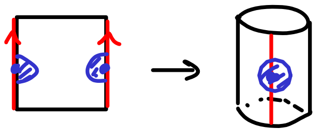
Ex: $X = \mathbb{R}^{n+1} - \{0\}$, define an equivalence relation $x \sim y$ iff x, y lie on the same line through the origin, ie. $x = \alpha y$ for some $\alpha \in \mathbb{R}$, $\alpha \neq 0$. This is an equivalence relation.
The quotient space is projective n-space, $\mathbb{RP}^n = X/\sim$ with quotient topology.
("space of lines through 0 in $\mathbb{R}^{n+1}$")

If Y is another top space, then a continuous map $\bar f: \mathbb{RP}^n \to Y$ is the same thing as a continuous map $f: \mathbb{R}^{n+1} - \{0\} \to Y$ s.t. $f(\alpha x) = f(x)$ $\forall \alpha \in \mathbb{R} - \{0\}$, $\forall x \in X$.
(more about $\mathbb{RP}^n$ on the HW)

Ex: Various quotients of the unit square $X = [0,1]^2$: let the edges be $A = \{0\} \times [0,1]$, A', B, B'

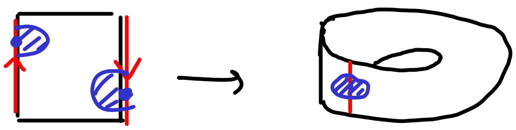


1) gluing A to A' by $(0,t) \sim (1,t)$, get a cylinder

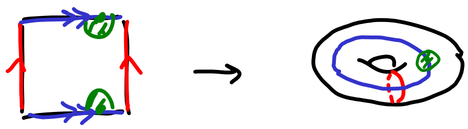


A neighborhood of a point on the gluing line corresponds to two neighborhoods of $(0,t) \in A$ and $(1,t) \in A'$ in X.

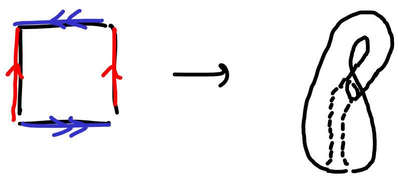
2) if instead we glue A to A' by $(0,t) \sim (1, 1-t)$, we get a <u>Möbius band</u> !



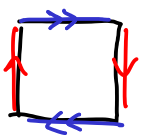
3) gluing A to A' via $(0,t) \sim (1,t)$ <u>and</u> B to B' by $(s,0) \sim (s,1)$ gives us the torus



4) gluing $(0,t) \sim (1,t)$ and $(s,0) \sim (1-s,1)$, however, gives the <u>Klein bottle</u>, which cannot be embedded in $\mathbb{R}^3$ (can draw a picture that self-intersects).



5) gluing $(0,t) \sim (1,1-t)$ and $(s,0) \sim (1-s,1)$ is tricky to visualize, but the quotient is actually homeomorphic to $\mathbb{R}P^2$.



(<u>Exercise</u>: what about gluing $(0,t) \sim (t,0)$ and $(1,s) \sim (s,1)$ – what does that look like ?).