Math 55b Lecture 10 - Friday Feb 19 - Algebraic topology! → Homotopy & fundamental group

<u>Recall</u>: given a surjective map $f: X \twoheadrightarrow A$ (eg. $A = X/\sim$), the <u>quotient topology</u> on A has $U \subset A$ open $\iff$ $f^{-1}(U) \subset X$ open. Say f is a <u>quotient map</u>

<u>Ex</u>: $[0,1] \twoheadrightarrow S^1$, $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ is a quotient map, ie. $[0,1]/(0\sim 1) \simeq S^1$.

★ There is a useful characterization of continuous maps from a quotient space: if $A = X/\sim$ and $f: X \to Y$ is a map st. $x \sim x' \Rightarrow f(x) = f(x')$, then we can define $\bar{f}: X/\sim \to Y$ by $\bar{f}([x]) = f(x)$. (& conversely)

$$\begin{tikzcd} & X/\sim \arrow[dr, "\bar{f}"] & \\ X \arrow[ur, "p"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

<u>Thm</u>: If $f: X \to Y$ is a continuous map and $x \sim x' \Rightarrow f(x) = f(x')$, then equipping $X/\sim$ with the quotient topology, $\bar{f}: X/\sim \to Y$ is a continuous map.

<u>Pf</u>: let $p: X \to X/\sim$, $x \mapsto [x]$ the quotient map, and recall $\bar{f}([x]) = f(x)$ (indep. of $x \in [x]$).

So $\bar{f} \circ p = f$. Hence: $\forall U \subset Y$ open, $f^{-1}(U) = p^{-1}(\bar{f}^{-1}(U)) \subset X$ is open.

By definition of the quotient topology, we conclude that $\bar{f}^{-1}(U) \subset X/\sim$ is open. □
($V \subset X/\sim$ open $\iff p^{-1}(V) \subset X$ open).

(& conversely, since $p: X \to X/\sim$ is continuous. So: $\bar{f}$ continuous $\iff f = \bar{f} \circ p$ continuous)

<u>Ex</u>: $X = \mathbb{R}^{n+1} - \{0\}$, define an equivalence relation $x \sim y$ iff x, y lie on the same line through the origin, ie. $x = \alpha y$ for some $\alpha \in \mathbb{R}$, $\alpha \neq 0$. This is an equivalence relation.

The quotient space is <u>projective n-space</u>, $\mathbb{RP}^n = X/\sim$ with quotient topology.
(<u>picture</u> $\mathbb{RP}^1$? $\mathbb{RP}^2$?) ("space of lines through 0 in $\mathbb{R}^{n+1}$")

If Y is another top space, then a continuous map $\bar{f}: \mathbb{RP}^n \to Y$ is the same thing as a continuous map $f: \mathbb{R}^{n+1} - \{0\} \to Y$ st. $f(\alpha x) = f(x)$ $\forall \alpha \in \mathbb{R} - \{0\}$, $\forall x \in X$.
(more about $\mathbb{RP}^n$ on the HW)

<u>Homotopy</u> = notion of continuous deformation, parametrized by $I = [0,1]$. (Munkres §51)

<u>Def</u>: $f, g: X \to Y$ two continuous maps. A <u>homotopy</u> between f and g is a continuous map $H: X \times I \to Y$ st. $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ $\forall x \in X$.
(the "time" variable in the homotopy)

If this exists, then say f and g are <u>homotopic</u> and write $f \simeq g$.

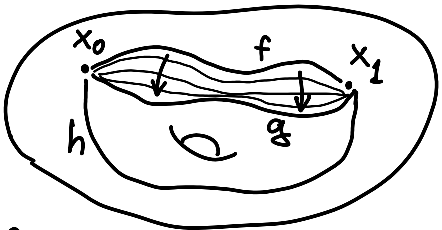
If f is homotopic to a constant map, we say it is <u>nullhomotopic</u>.

We'll want to study <u>paths</u> in top spaces, ie $f: [0,1] \to X$ continuous, $f(0) = x_0$, $f(1) = x_1$.

The above notion is not useful for paths if we don't fix the end points x_0 & x_1 (see HW4).

Better notion: <u>homotopy of paths</u> only considers homotopies which keep the end points in place.

(General notion: pairs (X,A) $A\subset X$ subspace, maps of pairs $(X,A)\xrightarrow{f}(Y,B)$: $f(A)\subset B$)



$f\simeq_p g$ homotopic paths
h not homotopic to f & g.

Def: Two paths $f,g: I\to X$ from x_0 to x_1 are <u>(path) homotopic</u> if $\exists$ continuous $H: I\times I\longrightarrow X$ st. $H(s,0)=f(s)$, $H(s,1)=g(s)$ (homotopy) and $H(0,t)=x_0$, $H(1,t)=x_1$ (fix end points: so $\forall t\in[0,1]$, $f_t=H_{|I\times\{t\}}$ is a path from x_0 to x_1)

Such H is a <u>path homotopy</u>, and we write $f\simeq_p g$.

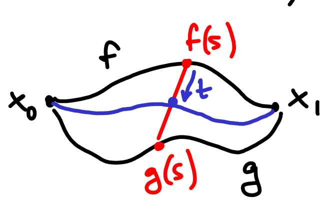
Lemma: $\simeq$ and $\simeq_p$ are equivalence relations.

<u>Pf</u>:
- clearly $f\simeq f$ (constant homotopy $H(x,t)=f(x)$).
- if $f\simeq g$ with homotopy $F(x,t)$, then the <u>reverse</u> homotopy $G(x,t)=F(x,1-t)$ gives $g\simeq f$.
- Assume $f\simeq g$ with homotopy $F(x,t)$
  $g\simeq h$ —"— $G(x,t)$
  then the <u>concatenation</u> of these is

  $$H: X\times[0,1]\longrightarrow Y \text{ defined by } H(x,t)=\begin{cases}F(x,2t) & \text{if } t\in[0,\frac{1}{2}]\\ G(x,2t-1) & \text{if } t\in[\frac{1}{2},1]\end{cases}$$

  These two formulas agree at $t=\frac{1}{2}$ $(F(x,1)=g(x)=G(x,0))$ so H is well-defined and continuous (cf "pasting lemma" Thm 18.3), and gives a homotopy $f\simeq h$.
- In the case of path homotopies, can check the above constructions preserve the requirements $F(0,t)=x_0$ & $F(1,t)=x_1$, so yield path homotopies. □

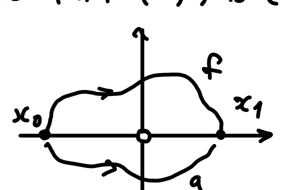
<u>Ex</u>: 1) If f,g are paths in $\mathbb{R}^n$ (or any convex subset of $\mathbb{R}^n$) from x_0 to x_1, we can define the <u>straight-line homotopy</u> $F(s,t)=(1-t)f(s)+tg(s)$



For each s, this connects $f(s)$ to $g(s)$ by a straight line segment. We conclude: $f\simeq_p g$ always!

2) in the punctured plane $X=\mathbb{R}^2-\{(0,0)\}$, let f,g be paths from $(-1,0)$ to $(1,0)$ st. f stays in the upper half plane $\{(x,y)\mid y\geq 0\}$
g —"— lower —"— $y\leq 0$

Then there is no homotopy between f & g in X.
(we'll prove this rigorously later).



Def: Spaces X, Y are <u>homotopy equivalent</u> if $\exists\ f: X \to Y,\ g: Y \to X$ st. $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$ (3)

(Check: this is an equiv^{ce} relation)

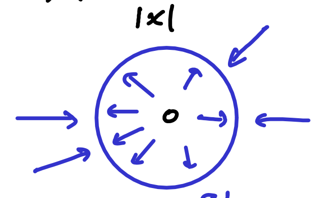
homotopic (vs. exact inverse would be homeo).

Def: X is <u>contractible</u> if X is homotopy eq. to $\{point\}$.

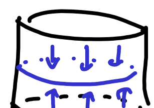
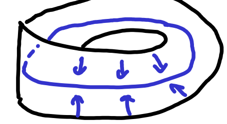
Ex: • $\mathbb{R}^n$ (or a convex subset of $\mathbb{R}^n$) is contractible: ie. $\{0\} \underset{r: x \mapsto 0}{\overset{i}{\rightleftarrows}} \mathbb{R}^n$
check $i \circ r$ = zero map is homotopic to $id_{\mathbb{R}^n}$ by $H(x,t) = tx$.

Ex: • $\mathbb{R}^2 - \{0\}$ is not contractible, but homotopy eq. to S^1, via $S^1 \overset{i}{\rightleftarrows} \mathbb{R}^2 - \{0\}$, $r: x \mapsto \frac{x}{|x|}$.
r is a <u>deformation retraction</u> of $X = \mathbb{R}^2 - \{0\}$ onto its subset $A = S^1 \subset X$, ie. $\begin{cases} \bullet\ r: X \to A \\ \bullet\ r_{|A} = id_A \quad (\text{ie. } r \circ i = id_A) \\ \bullet\ i \circ r: X \to A \subset X \text{ is } \simeq id_X. \end{cases}$

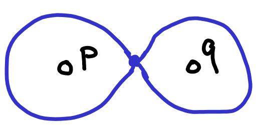
(this is called a retraction; less useful!)
(the "deformation")

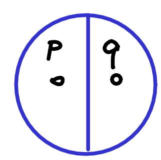


(in this case, $i \circ r(x) = \frac{x}{|x|}$ homotopic to id by straight line homotopy)
Deformation retraction is a useful special case of homotopy equiv^{ce}

By the same argument the cylinder $S^1 \times I$  & Möbius band 
deformation retract onto "middle" S^1 by sliding points of $[0,1]$ to midpoint.
(<u>check</u>: this is consistent with the twisted gluing of $I \times I$, $(0,y) \sim (1, 1-y)$).
hence they are homotopy equivalent to S^1 (and to each other and to $\mathbb{R}^2 - \{0\}$).

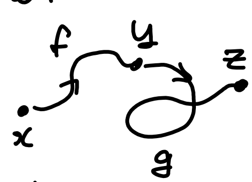
<u>Ex</u>: $\mathbb{R}^2 - \{p,q\}$ deformation retracts onto wedge of two S^1's ("figure 8" space)



(or also on "theta graph" (htpy eq. to ∞, not homeo!) )

• We now focus on paths and path-homotopy as a way to define an algebraic invt of top. spaces (up to homotopy equiv^{ce}): the <u>fundamental group</u>. A group needs a multiplication?

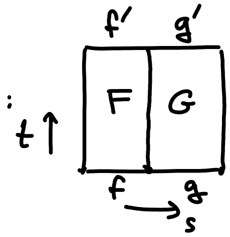
Def. if f is a path from x to y and g is a path from y to z, define a path $f * g$ from x to z by running through first f then g (twice as fast):
$$(f*g)(s) = \begin{cases} f(2s) & \text{if } s \in [0, \frac{1}{2}] \\ g(2s-1) & \text{if } s \in [\frac{1}{2}, 1] \end{cases}$$



This product is well-defined on path-homotopy classes, as long as $f(1) = g(0)$:
if $f \simeq_p f'$ and $g \simeq_p g'$ then $f * g \simeq_p f' * g'$

using homotopy $(F*G)(s,t) = \begin{cases} F(2s,t) & s \le \frac{1}{2} \\ G(2s-1,t) & s \ge \frac{1}{2} \end{cases}$.

Picture:



So we define $[f]*[g] = [f*g]$

<u>Claim</u>: this operation is <u>associative</u>, and has <u>identity</u> & <u>inverses</u>.

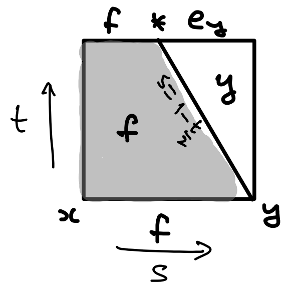
⇝ the "<u>fundamental groupoid</u>" of X: category with objects = points of X and $\mathrm{Mor}(x,y) = \{$path homotopy classes of paths $x \to y\}$.

$\begin{cases} \text{category: composition is associative} + \exists \text{ identity morphisms } x \to x \\ \text{groupoid: all morphisms have inverses.} \end{cases}$

★ <u>Identity</u>: given $x \in X$, consider the constant path $e_x : I \to X$, $e_x(s) = x \;\forall s$, & let $id_x = [e_x]$.

We claim that if f is any path from x to y, then $[f]*id_y = id_x * [f] = [f]$.

Indeed, there are explicit homotopies $f \simeq_p (f*e_y)$ & similarly, $(e_x * f) \simeq_p f$.



$$F(s,t) = \begin{cases} f\left(\frac{s}{1-t/2}\right) & s \in [0, 1-\frac{t}{2}] \\ y & s \in [1-\frac{t}{2}, 1] \end{cases}$$

★ <u>Inverse</u>: given a path f from x to y, define the reverse path $\bar{f}(s) = f(1-s)$ from y to x.

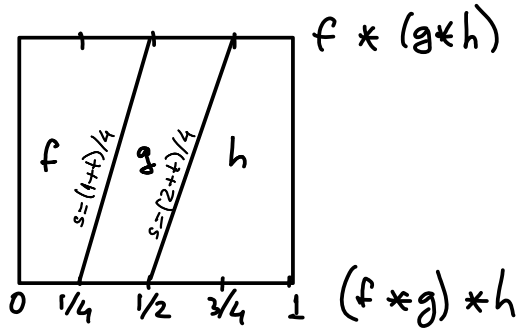
$[\bar{f}]$ is inverse to $[f]$, namely $e_x \simeq_p f * \bar{f}$ and $e_y \simeq_p \bar{f} * f$. Indeed:

$$F(s,t) = \begin{cases} f(2ts) & s \in [0, \frac{1}{2}] \\ f(2t(1-s)) & s \in [\frac{1}{2}, 1] \end{cases}$$

For given t this runs forward along f from $f(0) = x$ to $f(t)$ at $s = \frac{1}{2}$ then backwards to $f(0) = x$ at $s = 1$. For $t = 0$ get e_x, $t = 1$ get $f * \bar{f}$.

(Similarly for $e_y \simeq_p \bar{f} * f$).

★ <u>Associativity</u>: given paths f, g, h with $f(1) = g(0)$ and $g(1) = h(0)$, claim $(f*g)*h \simeq_p f*(g*h)$. Both run along f then g then h, but with different parametrizations. The homotopy comes from adjusting for this:



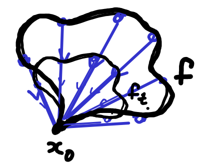
Let $F(s,t) = \begin{cases} f\left(\frac{4s}{1+t}\right) & s \in [0, \frac{1+t}{4}] \\ g(4s - (1+t)) & s \in [\frac{1+t}{4}, \frac{2+t}{4}] \\ h\left(\frac{4s-(2+t)}{2-t}\right) & s \in [\frac{2+t}{4}, 1]. \end{cases}$

Fundamental group: Groups are much easier to study than groupoids! want to be able to multiply always, not worrying whether end points match. Thus we fix a base point $x_0 \in X$ and only consider paths from x_0 to itself – ie. loops (based at x_0).

Def: The set of path homotopy classes of loops based at x_0, with operation $*$ (concatenation), is called the fundamental group of X, denoted $\pi_1(X, x_0)$.

Ex: in $\mathbb{R}^n$ (or a convex domain in $\mathbb{R}^n$), every loop at x_0 is path-homotopic to the identity (ie. the constant path at x_0) by the straight-line homotopy

$$F(t,s) = (1-t)f(s) + t\,x_0$$



So $\pi_1(\mathbb{R}^n, x_0) = \{id\}$.

Def: X is simply-connected if $X \neq \emptyset$ is path-connected, and for $x_0 \in X$, $\pi_1(X, x_0) = \{1\}$.

This def. is sensible because π_1 is, up to isom., independent of choice of x_0 inside a path component of X (we'll see this next time)