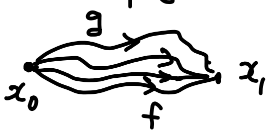
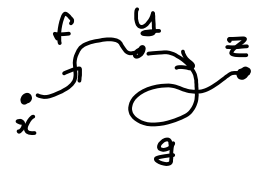
<u>Last time</u>: • paths $f, g : I = [0,1] \to X$ from $x_0$ to $x_1$ are <u>path-homotopic</u>, $f \simeq_p g$, if

$$\exists\, H : \underset{s,\,t}{I \times I} \to X, \quad H(s,0) = f(s) \quad H(0,t) = x_0$$
$$H(s,1) = g(s) \quad H(1,t) = x_1$$




• <u>composition</u> of paths $f$ from $x$ to $y$, $g$ from $y$ to $z$:

$$(f * g)(s) = \begin{cases} f(2s) & \text{if } s \in [0, \tfrac{1}{2}] \\ g(2s-1) & \text{if } s \in [\tfrac{1}{2}, 1] \end{cases}$$




• This product is well-defined on path-homotopy classes, as long as $f(1) = g(0)$: if $f \simeq_p f'$ and $g \simeq_p g'$ then $f * g \simeq_p f' * g'$. Define $[f] * [g] = [f * g]$.

• The operation $*$ on path-homotopy classes is <u>associative</u>, and has <u>identity</u> & <u>inverses</u>,

identity: $\forall x \in X$, $e_x$ = constant path at $x$, $f(s) = e_x$.

inverse: $\bar{f}(s) = f(1-s)$ reverse path. given $f$ from $x$ to $y$, $f * \bar{f} \simeq_p e_x$, $\bar{f} * f \simeq_p e_y$.

associative: if $f(1) = g(0)$ & $g(1) = h(0)$, $(f * g) * h \simeq_p f * (g * h)$.

---

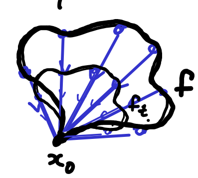
To get a group out of this, we fix a <u>base point</u> $x_0 \in X$ and only consider <u>loops</u> based at $x_0$, ie. paths from $x_0$ to itself

<u>Def</u>: The set of path-homotopy classes of loops based at $x_0$, with operation $*$, is called the <u>fundamental group</u> of $X$, denoted $\pi_1(X, x_0)$. (<u>check</u>: it <u>is</u> a group)

<u>Ex</u>: in $\mathbb{R}^n$ (or a convex domain in $\mathbb{R}^n$), every loop at $x_0$ is path-homotopic to the identity (ie. the constant path at $x_0$) by the straight-line homotopy

$$F(t,s) = (1-t) f(s) + t\, x_0$$

So $\pi_1(\mathbb{R}^n, x_0) = \{id\}$.




<u>Def</u>: $X$ is <u>simply-connected</u> if $X \neq \emptyset$ is path-connected, and for $x_0 \in X$, $\pi_1(X, x_0) = \{1\}$.

<u>Ex</u>: we'll see at some point: $\pi_1(S^1, x_0) \simeq \mathbb{Z}$ ("# turns of a loop around the circle")

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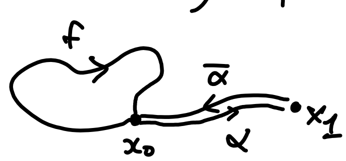
* Dependence on the base point:

If $x_0, x_1$ are in the same path-component of $X$, let $\alpha$ be a path from $x_0$ to $x_1$.

Then for any loop $f$ based at $x_0$, we get a loop at $x_1$ by taking $\bar{\alpha} * f * \alpha$,




and so we get a map $\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$

$$[f] \longmapsto [\bar{\alpha} * f * \alpha] = [\bar{\alpha}] * [f] * [\alpha]$$

(recall: $*$ well-def'd on path-homotopy classes).

<u>Prop</u>: $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is a <u>group isomorphism</u>.

Proof. • if $a, b \in \pi_1(X, x_0)$ then
$$\begin{aligned}\hat\alpha(a*b) &= [\bar\alpha]^{-1} * (a*b) * [\alpha] \\ &= [\bar\alpha] * a * [\alpha] * [\bar\alpha] * b * [\alpha] \quad \text{(using associativity \& inverses)} \\ &= \hat\alpha(a) * \hat\alpha(b).\end{aligned}$$
So $\hat\alpha$ is a group homomorphism.

• let $\beta = \bar\alpha$ reverse path from $x_1$ to $x_0$, then $\hat\beta : \pi_1(X, x_1) \to \pi_1(X, x_0)$.
We claim $\hat\beta$ and $\hat\alpha$ are inverses of each other. Indeed: for $a \in \pi_1(X, x_0)$,
$$\begin{aligned}\hat\beta(\hat\alpha(a)) = \hat\beta([\bar\alpha] * a * [\alpha]) &= [\beta] * [\bar\alpha] * a * [\alpha] * [\beta] \\ &= [\alpha] * [\bar\alpha] * a * [\alpha] * [\bar\alpha] = a.\end{aligned}$$
Hence $\hat\beta \circ \hat\alpha = id$ (and similarly $\hat\alpha \circ \hat\beta = id$ as well), so $\hat\alpha$ is an isomorphism. □

Corollary: if $X$ is path-connected, then $\pi_1(X, x_0)$ is independent of $x_0$ up to isomorphism.

Rmk: when $\alpha$ is a loop at $x_0$, we get an automorphism $\hat\alpha$ of $\pi_1(X, x_0)$. This is in fact an inner automorphism = conjugation by $[\alpha]$: $a \mapsto [\alpha]^{-1} * a * [\alpha]$.

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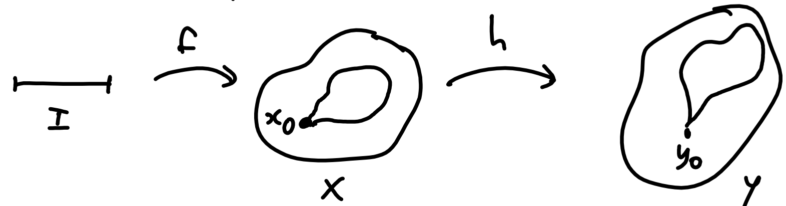
* $\pi_1$ as a functor: Consider the category of pointed topological spaces:

- objects = top. space + choice of base point, $(X, x_0)$
- morphisms = continuous maps preserving base points: $f : (X, x_0) \to (Y, y_0)$ means $f : X \to Y$ continuous & s.t. $f(x_0) = y_0$.

Def/Prop: A continuous map $h : (X, x_0) \to (Y, y_0)$ induces a group homomorphism $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by $h_*([f]) = [h \circ f]$.

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[rr, bend right, "h \circ f"'] & X \arrow[r, "h"] & Y \end{tikzcd}$$




Check: • if $f \simeq_p f'$ via $F$ then $h \circ f \simeq_p h \circ f'$ via $h \circ F$. So $h_*$ is well-defined.
• $h \circ (f * g) = (h \circ f) * (h \circ g)$ (composition w/ h compatible with concatenation)
So $h_*$ is a group homomorphism, $h_*([f] * [g]) = h_*([f]) * h_*([g])$.

Prop: given $(X, x_0) \xrightarrow{h} (Y, y_0) \xrightarrow{k} (Z, z_0)$, $(k \circ h)_* = k_* \circ h_* : \pi_1(X, x_0) \to \pi_1(Z, z_0)$.
hence $\pi_1$ is a functor (maps composition $k \circ h$ to composition $k_* \circ h_*$).

(this is just: $(k \circ h) \circ f = k \circ (h \circ f)$).

This implies: Corollary: if $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then $h_*$ is an isomorphism.

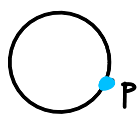
But we can do better!

**Recall:** • a <u>retraction</u> of $X$ onto a subset $A \overset{i}{\hookrightarrow} X$ is $r: X \to A$ st. $r_{|A} = id_A$, ie. $r \circ i = id_A$. Then, taking a base point $a_0 \in A$,

$$\pi_1(A, a_0) \underset{r_*}{\overset{i_*}{\rightleftarrows}} \pi_1(X, a_0) \qquad r_* \circ i_* = id \Rightarrow Ker(i_*) = \{1\}, \text{ ie. } i_* \text{ injective}$$

• a <u>deformation retraction</u> : assume moreover that $i \circ r : X \to X$ is homotopic to $id_X$ by a homotopy that fixes $A$. Then we claim $i_*, r_*$ are inverse isom's. $\pi_1(A, a_0) \simeq \pi_1(X, a_0)$.

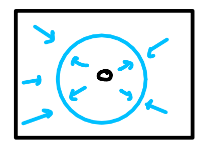
<u>Ex:</u>




$S^1 \to p$
constant map

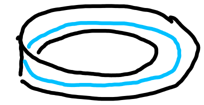

$S^2 \to S^2_+$
$(x,y,z) \mapsto (x,y,|z|)$

retractions, $i \circ r \neq id_X$




$\mathbb{R}^2 - \{0\} \to S^1$
$x \mapsto x/|x|$

deformation retractions



Möbius band $\to S^1$

• More generally, recall a <u>homotopy equivalence</u> is $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ st. $f \circ g \simeq id_Y$, $g \circ f \simeq id_X$.

<u>Thm:</u> Homotopy equivalences induce isomorphisms $\pi_1(X, x_0) \xrightarrow[f_*]{\sim} \pi_1(Y, f(x_0))$

This follows from the fact that homotopic maps induce the same homomorphisms on $\pi_1$, namely:

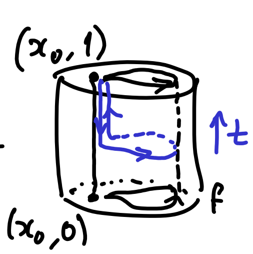
<u>Prop:</u> (1) Let $h, k : X \to Y$ homotopic via a homotopy $H: X \times I \to Y$ st. $H(x_0, t) = y_0$ $\forall t$. Then $h_* = k_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

(2) If the homotopy $H$ doesn't fix base points, let $\alpha$ be the path $y_0 \to y_1$ def^d by $\alpha(t) = H(x_0, t) = y_t$. Then $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, $k_* : \pi_1(X, x_0) \to \pi_1(Y, y_1)$ are related by the isom. $\hat{\alpha} : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ : $k_* = \hat{\alpha} \circ h_*$.

<u>Pf:</u> (1) given a loop $f: I \to X$ based at $x_0$,
$$I \times I \xrightarrow{f \times id} X \times I \xrightarrow{H} Y$$
$$(s,t) \mapsto (f(s), t) \mapsto H(f(s), t)$$
$H \circ (f \times id) : I \times I \to Y$ gives a path homotopy (based at $y_0$) $h \circ f \simeq_p k \circ f$, hence $h_*([f]) = k_*([f])$.

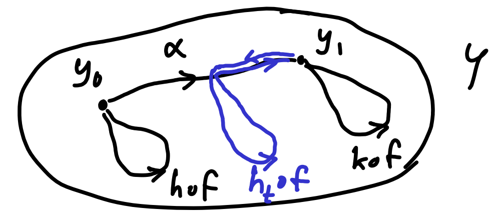
(2) now consider $I \times I \xrightarrow[F]{} X \times I$ def^d by concatenating $\begin{cases} \text{path } (x_0, 1) \to (x_0, t) \\ \text{loop } f \text{ in } X \times \{t\} \\ \text{path } (x_0, t) \to (x_0, 1). \end{cases}$

Then $H \circ F$ is a path homotopy in $(Y, y_1)$ from $\alpha^{-1} * (h \circ f) * \alpha$ to $e * (k \circ f) * e$. □




$F$ path homotopy in $(X \times I, (x_0, 1))$

$\xrightarrow{H}$



→ <u>Pf-thm:</u> if $(X,x_0) \xrightarrow{f} (Y,y_0) \xrightarrow{g} (X,x_1)$ homotopy inverses, $g\circ f \simeq id_X$

⇒ by the prop$^n$,

$$\begin{tikzcd} \pi_1(X,x_0) \arrow[r, "f_*"] \arrow[rr, bend right, "(g\circ f)_*"'] & \pi_1(Y,y_0) \arrow[r, "g_*"] \arrow[rr, bend left, "(f\circ g)_*"] & \pi_1(X,x_1) \arrow[r, "f'_*"] & \pi_1(Y,y_1) \end{tikzcd}$$

$(g\circ f)_* = \hat{\alpha}$ for some path $\alpha : x_0 \rightsquigarrow x_1$

⇒ this is an isom.

Hence $f_*$ is injective & $g_*$ is surjective.

Similarly, $(f\circ g)_*$ isom. $\pi_1(Y,y_0) \to \pi_1(Y,y_1)$ ⇒ $g_*$ injective, $f'_*$ surjective.

Hence $g_*$ is an iso, and $f_* = (g_*)^{-1}\circ\hat{\alpha}$ is also an isom. □

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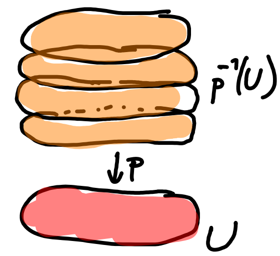
At some point we'd like to show $\pi_1(S^1) \cong \mathbb{Z}$. We'll do this by introducing a key tool for the study of $\pi_1$: the notion of <u>covering spaces</u>.

<u>Def:</u> Let $p : E \to B$ be a continuous surjective map. We say $p$ <u>evenly covers</u> an open subset $U \subset B$ if $p^{-1}(U) = \bigcup_{\alpha\in A} V_\alpha$ where $V_\alpha \subset E$ are disjoint open subsets, and for each $\alpha \in A$, $p_{|V_\alpha} : V_\alpha \to U$ is a homeomorphism. The $V_\alpha$ are called <u>slices</u>.



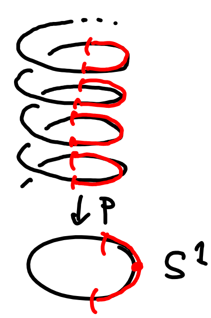

(equivalently: $\exists\ p^{-1}(U) \xrightarrow[\varphi]{\text{homeo} \sim} U \times A$ discrete top. st. $p_{|U} = pr_1 \circ \varphi$).

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "p_{|U}"'] & & U\times A \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

say diagram of maps "<u>commutes</u>".

<u>Def:</u> If every point of $B$ has a neighborhood which is evenly covered by $p$, we say $E$ is a <u>covering space</u> of $B$ and $p$ is a <u>covering map</u>. $B$ is called the <u>base</u> of the covering.

<u>Ex:</u> define $p : \mathbb{R} \to S^1$
$p(t) = (\cos t, \sin t)$




This is a covering map! for instance consider $(1,0) \in S^1$ and the neighborhood $U = \{(x,y) \in S^1 \mid x > 0\}$.

Then $p^{-1}(U) = \bigsqcup_{n\in\mathbb{Z}} \left(2\pi n - \frac{\pi}{2}, 2\pi n + \frac{\pi}{2}\right)$ and $p$ is a homeo. on each slice.

• <u>Thm:</u> $p : E \to B$, $q : E' \to B'$ covering maps ⇒ $p\times q : E\times E' \to B \times B'$ is a covering map.

<u>Pf:</u> given $(b,b') \in B\times B'$, let $U \ni b$ and $U' \ni b'$ be neighborhoods st.

$p^{-1}(U) = \bigsqcup V_\alpha$, $q^{-1}(U') = \bigsqcup V'_\beta$ slices, then

$(p\times q)^{-1}(U\times U') = p^{-1}(U) \times q^{-1}(U') = \bigsqcup_{\alpha,\beta} V_\alpha \times V'_\beta$ union of open slices homeo to $U\times U'$! □