Math 55b Homework 5

Due Wednesday March 3, 2021.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.
- Questions marked * may be on the harder side.

Material covered: fundamental group, covering spaces, Brouwer fixed point theorem, homotopy equivalence. (Munkres §52-55, 58).

0. Sometime over the weekend of February 27-28, please complete the week 5 feedback survey (in Canvas). This is important to help us assess how well the course structure, pacing, and our efforts at getting students to know each other are working. (There will be more surveys).

1. Munkres exercise 52.4.

2. Munkres exercise 52.5.

3. Munkres exercise 53.3.

4. Munkres exercise 53.4.

5. The Möbius band B is the quotient of $[0,1]\times[0,1]$ by the relation $(0,y)\sim(1,1-y)$. Show that the cylinder $S^1\times[0,1]$ is a degree 2 (that is, 2-sheeted) covering space of B.

6. Munkres exercise 54.8.

7. Munkres exercise 55.1.

8. Munkres exercise 55.2.

9. Munkres exercise 58.2.

10. Let G be a *topological group*, i.e. a group (with operation $\cdot$ and identity element $x_0\in G$) which is also a topological space, in such a way that the multiplication map $\cdot : G\times G\to G$ and the map $G\to G$ taking each element to its inverse are continuous. Examples of topological groups include: $(\mathbb{R},+)$, $(S^1,\cdot)$, $GL(n,\mathbb{R})$, $SO(3)$, etc.

Given two loops $f,g: I\to G$ based at x_0, we define a loop $f\cdot g$ by $(f\cdot g)(s)=f(s)\cdot g(s)$.

(a) Show that this operation makes the set of based loops in (G,x_0) into a group, and that it induces a group operation $\cdot$ on the set of path-homotopy classes $\pi_1(G,x_0)$.

(b) Show that the two group operations $*$ and $\cdot$ on $\pi_1(G,x_0)$ are the same. (Hint: consider $(f*e_{x_0})\cdot(e_{x_0}*g)$).

(c) Show that $\pi_1(G,x_0)$ is abelian.

11. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?