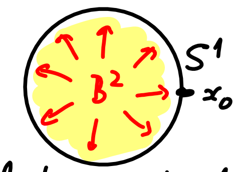
The Brouwer fixed point theorem:

Let B^n denote the closed ball of radius 1 in $\mathbb{R}^n$, with boundary the unit sphere S^{n-1}.
Recall that, if $A \subset X$, a retraction $r: X \to A$ is a continuous map s.t. $r(a) = a \ \forall a \in A$.

Thm: There is no retraction of B^2 onto S^1.



Pf: if $r: B^2 \to S^1$ is a retraction, then $i \circ r = id_{S^1}$, so

$$\underset{\mathbb{Z}}{\pi_1(S^1, x_0)} \xrightarrow{i_*} \underset{\{1\}}{\pi_1(B^2, x_0)} \xrightarrow{r_*} \pi_1(S^1, x_0) \qquad i_* \circ r_* = \text{trivial hom.} \neq id: \mathbb{Z} \to \mathbb{Z}.$$

$\{1\}$ (convex $\subset \mathbb{R}^2$, straight line homotopy) Contradiction. □

(More elementary way to say this: given a nontrivial loop f in S^1, $i \circ f$ is nullhomotopic in B^2, via some homotopy H from f to e_{x_0}. Then $r \circ H$ is a path-homotopy $f \rightsquigarrow e_{x_0}$ in S^1, contradiction.).

[with more alg. top., similarly $\nexists$ retraction $B^n \to S^{n-1}$ $\forall n$].

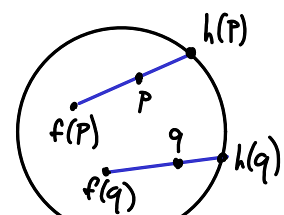
$\Rightarrow$ **Brouwer fixed point theorem:**

If $f: B^2 \to B^2$ is continuous, then $\exists x \in B^2$ st. $f(x) = x$.

(case $n=1$ follows from intermediate value thm, cf. HW2)

[with more alg. top., the same holds for continuous maps $B^n \to B^n$ $\forall n$.]

Proof: assume $f: B^2 \to B^2$ continuous, $f(x) \neq x \ \forall x \in B^2$.
Then define $h: B^2 \to S^1$ by mapping each $p \in B^2$ to the point where the ray from $f(p)$ to p hits $\partial B^2 = S^1$.



(formula: $h(p) = p + t(p - f(p))$ where $t > 0$ st. $\|h(p)\|^2 = 1$.
can solve by quadratic formula, so t does depend continuously on p).
This gives a continuous map $h: B^2 \to S^1$, moreover if $p \in S^1$ then $h(p) = p$,
so we get a retraction $B^2 \to S^1$. Contradiction. □

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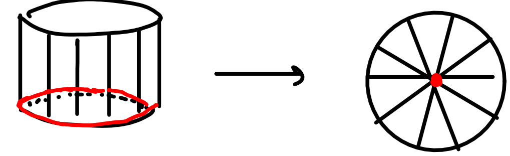
* A loop in (X, x_0) is defined as a map $I \to X$ st. $\{0,1\} \to \{x_0\}$, but since $I/0\sim 1$ is homeo. to S^1, can also think of it as a map $(S^1, p_0) \xrightarrow{f} (X, x_0)$.
So $\pi_1(X, x_0)$ tells us about homotopy classes of maps $(S^1, p_0) \to (X, x_0)$... but also $S^1 \to X$.

Lemma: Let $h: S^1 \to X$ continuous, then the following are equivalent:
(1) h is nullhomotopic
(2) h extends to a continuous map $k: B^2 \to X$ $\quad (k_{|\partial B^2 = S^1} = h)$.
(3) $h_*: \pi_1(S^1) \to \pi_1(X)$ is the trivial homomorphism.

Pf: $(1)\Rightarrow(2)$ key observ^{n}: $\begin{array}{rcl} S^1\times I & \xrightarrow{p} & B^2 \\ (x,t) & \mapsto & t\cdot x \end{array}$ is a quotient map ie. $B^2 \simeq S^1\times I/(x,0)\sim(x',0)\ \forall x,x'$. ②

So: given a homotopy $H: S^1\times I \to X$ between a constant map and $h: S^1\to X$, $H(x,0)=H(x',0)\ \forall x,x'\in S^1$

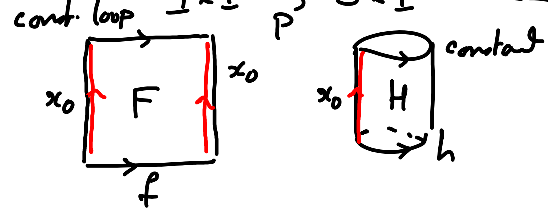


it factors through the quotient $S^1\times I \xrightarrow{p} B^2 \xrightarrow{\exists k} X$. In other terms: we can define $k: B^2\to X$ by $k(t\cdot x)=H(x,t)$ despite angular coordinate x not being well-defined at $t=0$, and k is continuous. By construction $k_{|S^1}=h$.

$(2)\Rightarrow(3)$: if $h=k_{|S^1}$ then can write $h=k\circ i$ where $i: S^1\hookrightarrow B^2$ is the inclusion. By functoriality of π_1, $h_*=k_*\circ i_*$: $\pi_1(S^1)\xrightarrow{i_*}\pi_1(B^2)\xrightarrow{k_*}\pi_1(X)$, composite h_*,
but $\pi_1(B^2)=\{1\}$, so k_* is trivial and so is h_*.

$(3)\Rightarrow(1)$: $h_*: \pi_1(S^1)\to\pi_1(X)$ trivial $\Rightarrow$ the loop $f: I\to X$ $s\mapsto h(e^{2\pi i s})$ ($=h\circ$(standard loop going around S^1)) represents the trivial elt of $\pi_1(X,x_0)$ ($x_0=h(1)$)
hence $\exists$ path-homotopy $F: I\times I\to X$ from f to constant loop at x_0; note that $F(0,t)=F(1,t)=x_0\ \forall t\in I$. Recall $I\times I/(0,t)\sim(1,t)\ \forall t$ is homeo. to $S^1\times I$.
↳ this implies F factors through the quotient:

H gives a homotopy from h to const. map. $I\times I \xrightarrow{p} S^1\times I \xrightarrow{\exists H} X$



□

(Ex: the inclusion $S^1\hookrightarrow\mathbb{R}^2-\{0\}$ and the identity map $S^1\to S^1$ aren't nullhomotopic, using lemma + i_* nontrivial on π_1)

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* Another application: <u>the fundamental thm. of algebra</u>

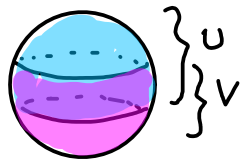
‖ $f(z)=z^d+a_{d-1}z^{d-1}+\dots+a_0$ complex polynomial of deg $d>0$ $\Rightarrow \exists z_0\in\mathbb{C}$ st. $f(z_0)=0$.

Pf: For $|z|=r>>0$, the term z^d dominates (as soon as $r^k>d|a_{d-k}|\ \forall 1\le k\le d$ so that $|a_{d-k}z^{d-k}|<\frac{1}{d}r^d$), so straight line segment $f(z)\to z^d$ doesn't cross 0.
$\Rightarrow F(z,t)=(1-t)f(z)+tz^d$ has no zeros on $\{|z|=r\}\times I$.
Hence: the maps $S^1\to S^1$ defined by $e^{i\theta}\mapsto\dfrac{f(re^{i\theta})}{|f(re^{i\theta})|}$ and $e^{i\theta}\mapsto e^{ni\theta}$ are homotopic via $(e^{i\theta},t)\mapsto F(re^{i\theta},t)/|F(re^{i\theta},t)|$.
These are nontrivial on $\pi_1(S^1)$ (in fact, map generator $1\in\mathbb{Z}$ to $d\in\mathbb{Z}_{>0}$) hence don't extend over B^2. But if f had no roots, $z\mapsto f(rz)/|f(rz)|$ would be such an extension.
□

Further study of π_1 - introduction to Seifert-Van Kampen

Q: Assume $X = U \cup V$, with U and V open subsets, and we know $\pi_1(U)$ and $\pi_1(V)$. Can we find $\pi_1(X)$?

Eg:



$S^2 = U \cup V$, $\pi_1(U)$ & $\pi_1(V)$ trivial

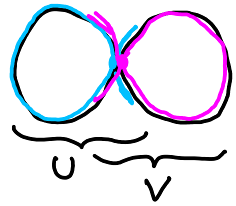


figure 8 $= U \cup V$, each of U & V has homotopy type of S^1.

The Seifert-Van Kampen, which we'll see soon, gives a general way to calculate $\pi_1(X)$ in this situation. For now we'll just prove a weaker (and easier) version.

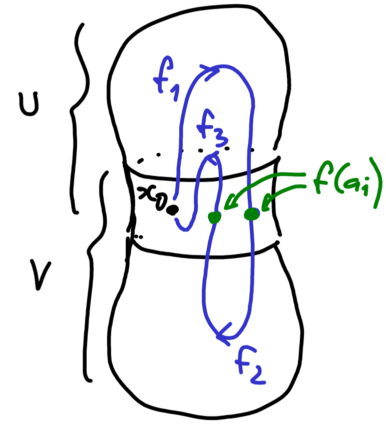
Thm: Suppose $X = U \cup V$, U and V open, $U \cap V$ path-connected, $x_0 \in U \cap V$. Let $i : U \hookrightarrow X$ and $j : V \hookrightarrow X$ be the inclusion maps. Then the images of $i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$ and $j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$ generate $\pi_1(X, x_0)$.

ie.: every element of $\pi_1(X, x_0)$ can be expressed as a product of elements in $\mathrm{Im}(i_*)$ and $\mathrm{Im}(j_*)$ - ie. every loop in (X, x_0) is path-homotopic to a composition of loops entirely contained in either U or V.

Pf: Let $f : I \to X$ be a loop based at x_0.

$[0,1] = f^{-1}(U) \cup f^{-1}(V)$ open cover, $[0,1]$ compact

$\Rightarrow$ using the Lebesgue number lemma, we can subdivide $[0,1]$ into $0 = a_0 < a_1 < \dots < a_n = 1$ st. $f([a_{i-1}, a_i])$ is contained in either U or V. Eliminating unnecessary a_i from the list, can assume U and V alternate along the way, and in particular $f(a_i) \in U \cap V$ $\forall i$.



Let $f_i = f_{|[a_{i-1}, a_i]}$ so that $[f] = [f_1] * \dots * [f_n]$.

For each i, choose a path α_i in $U \cap V$ from x_0 to $f(a_i)$. (take $\alpha_0 = \alpha_n =$ constant path at x_0).

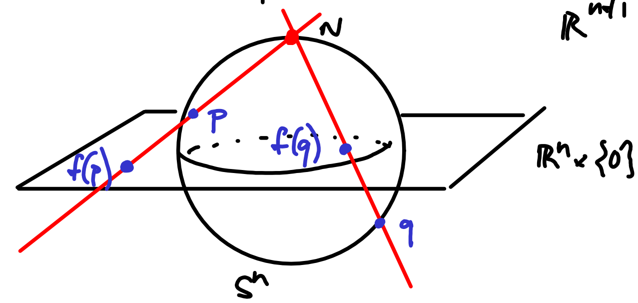
$$\text{Then } [f] = \underbrace{[\alpha_0 * f_1 * \alpha_1^{-1}] * [\alpha_1 * f_2 * \alpha_2^{-1}]}_{\text{loops at } x_0\text{, entirely contained in } U \text{ or in } V} * \dots * [\alpha_{n-1} * f_n * \alpha_n^{-1}]$$

$\square$

Corollary: $X = U \cup V$ with U & V open and simply-connected, $U \cap V$ path connected $\Rightarrow X$ is simply-connected.

Ex: Let $X = S^n$, $n \geq 2$, and $U = S^n - (0,0,\dots,0,1)$, $V = S^n - (0,\dots,0,-1)$
N: North pole, S: South pole.

Then U and V are homeomorphic to $\mathbb{R}^n$ via *stereographic projection* $f: U \to \mathbb{R}^n$ mapping each point $x \in U$ to the point where the line in $\mathbb{R}^{n+1}$ through N and x intersects the equatorial plane $\mathbb{R}^n \times \{0\}$.



I.e.: $f(x) = \dfrac{1}{1 - x_{n+1}} (x_1, \dots, x_n)$ (exercise: check this is a homeo.)

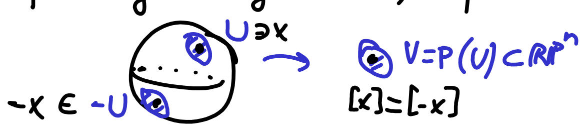
change to $+$ for $V \xrightarrow{\sim} \mathbb{R}^n$.

Hence: U and V, homeomorphic to $\mathbb{R}^n$, are simply connected
$U \cap V$ —"— $\mathbb{R}^n - \{\text{point}\}$, is path-connected ($n \geq 2$!)

Corollary: S^n is simply connected for $n \geq 2$.

$\Rightarrow$ **Corollary:** an open subset in $\mathbb{R}^{n\geq 3}$ cannot be homeomorphic to an open subset in $\mathbb{R}^2$.
Indeed: $U \subset \mathbb{R}^n$ open, $p \in U \Rightarrow \exists$ open ball $p \in B_r(p) \subset U$, and $B_r(p) - \{p\}$ deform. retracts onto a sphere $\Rightarrow B_r(p) - \{p\}$ is simply connected. Whereas $q \in V \subset \mathbb{R}^2$ open $\Rightarrow \forall$ open $q \in N \subset V$, $N - \{q\}$ can't be simply connected (retracts to circle).
(The argument for $\mathbb{R}^{n\geq 2}$ vs. $\mathbb{R}$ is easier, only uses connectedness)

Ex: recall from HW: the quotient of S^n by $x \sim -x$, $p: S^n \to S^n/\sim \; \simeq \mathbb{RP}^n$ is a degree 2 covering map.

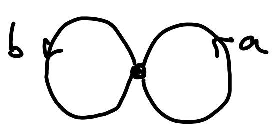


$p^{-1}(V) = U \sqcup (-U)$ ✓

Also recall: lifting correspondence $\pi_1(\mathbb{RP}^n, b_0) \longrightarrow p^{-1}(b_0) = \{2 \text{ points}\}$
surjective because S^n connected; injective because S^n is simply connected if $n \geq 2$
(if a loop f in $\mathbb{RP}^n$ lifts to a loop $\tilde{f}$ in S^n, then $\tilde{f}$ is homotopic to constant loop in S^n, & projecting by p, $p \circ \tilde{f} = f$ is homotopic to a constant loop in $\mathbb{RP}^n$).

For $n \geq 2$, $\pi_1(\mathbb{RP}^n)$ is a group with 2 elements, hence isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Ex: X = figure 8 space,

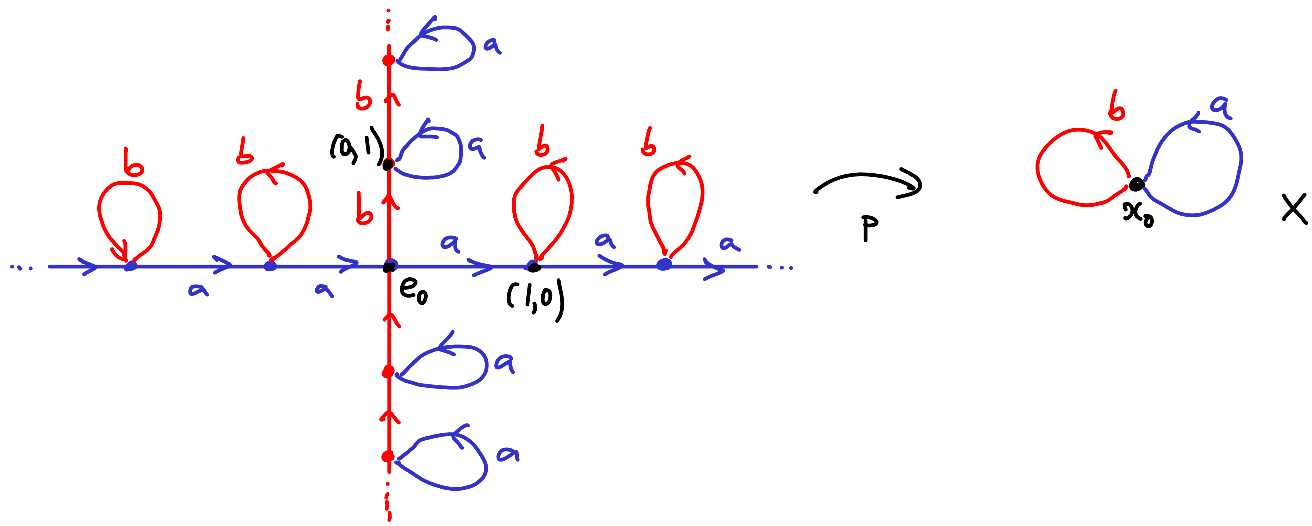


can cover by opens U, V which have deformation retractions to S^1, $U \cap V = \times$ connected
By theorem, $\pi_1(X)$ is generated by the images of two maps from $\mathbb{Z}$,
i.e. can express every loop in terms of powers of $[a]$ and $[b]$ (a, b loops around each S^1)
generators of $\pi_1(U)$, $\pi_1(V)$, i.e. every element is a product of $[a]^{\pm 1}$'s & $[b]^{\pm 1}$'s.

but don't know relations between $[a]$ and $[b]$.

Can show that $[a]$ and $[b]$ don't commute – $[a]*[b] \neq [b]*[a]$.

One way to do this is by looking at covering map



The lift of $a*b$ starting at e_0 ends at $(1,0)$ hence $[a]*[b] \neq [b]*[a]$

—— " —— $b*a$ —— " —— at $(0,1)$

so $\pi_1(X, x_0)$ is not abelian. In fact, we'll show later that it is the <u>free group</u> generated by $[a]$ and $[b]$, ie. elts are arbitrary words in $[a]^{\pm 1}$ and $[b]^{\pm 1}$ with no relations whatsoever (except $[a]^{-1}*[a] = 1$ etc.).