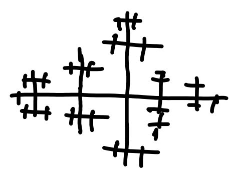
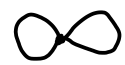
Last time:

- covering maps $p: E \to B$ (E and B path-connected & loc path conn^{d})
  induce an <u>injective</u> homomorphism $p_*: \pi_1(E, e_0) \to \pi_1(B, b_0)$, whose image
  $H = \mathrm{Im}(p_*) \subset \pi_1(B, b_0)$ (= those homotopy classes of loops in (B, b_0) which lift to loops in (E, e_0) (rather than just paths))
  determines the covering up to equivalence (= homeomorphism $E \xrightarrow[\sim]{h} E'$, $p \searrow B \swarrow p'$).
  Namely: given 2 coverings $\begin{cases} p: (E, e_0) \to (B, b_0) \\ p': (E', e_0') \to (B, b_0) \end{cases}$ & corresp. subgroups $H = \mathrm{Im}\, p_*$, $H' = \mathrm{Im}\, p'_*$:
- $\exists$ base point preserving equivalence ($h(e_0) = e_0'$) iff $H = H'$
- $\exists$ equivalence (not necess. mapping $e_0 \mapsto e_0'$) iff H, H' are conjugate subgroups of $\pi_1(B, b_0)$.

Universal covering space:

<u>Def</u>: If $p_0: E_0 \to B$ covering and E_0 is simply connected, say E_0 is a <u>universal covering</u> of B.

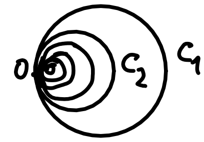
<u>Note</u>: this corresponds to the trivial subgroup $p_{0*}(\pi_1(E_0)) = \{1\} \subset \pi_1(B)$; unique up to equiv^{ce} by the above.

<u>Ex</u>:
- $p: \mathbb{R} \to S^1$
  $p \times p: \mathbb{R}^2 \to S^1 \times S^1 = $ torus
-  infinite tree $\longrightarrow$  $\left(\begin{array}{l}\text{horizontal} \\ \text{vertical}\end{array}\text{ edges} \to \begin{array}{l}a \\ b\end{array}\right)$.

- <u>Thm</u>: $p_0: E_0 \to B$ universal covering, $p': E' \to B$ any path-connected covering, then $\exists$ covering map $q_0: E_0 \to E'$ s.t. $p' \circ q_0 = p_0$; and q_0 is univ. covering of E'.

q_0 is constructed by lifting: $\begin{array}{ccc} & & E' \\ & \overset{q_0}{\nearrow} & \downarrow p' \\ E_0 & \xrightarrow[p_0]{} & B \end{array}$ ($\exists$ since $p_{0*}(\pi_1(E_0)) = \{1\} \subset p'_*(\pi_1(E'))$.
& can show it's a covering map as well.

So, in fact, if B has a universal covering, all other coverings can then be obtained as quotients!

- Some spaces have no universal covering!
  <u>Ex</u>: "Hawaiian earrings" $= \bigcup_{n \geq 1} C_n$ circles of radius $\frac{1}{n}$ centered at $(\frac{1}{n}, 0)$ in $\mathbb{R}^2$ 

  Any covering space must evenly over a neighborhood of the origin, which prevents it from being simply connected. (for n suff^{tly} large, loop around C_n lifts to a loop).

- If one avoids such pathological examples - assuming B is (semi) locally simply connected, can build univ. cover as space of pairs (b, γ) where $\begin{cases} b \in B \\ \gamma = \text{homotopy class of path } b_0 \to b \end{cases}$

This has a preferred topology for which any simply conn^d nbd $U \ni b$ is evenly covered: if $b' \in U$, adding a path $b \rightsquigarrow b'$ inside U or its inverse gives a preferred bijection $\{$htpy classes of paths $b_0 \to b\} \longleftrightarrow \{$htpy classes of paths $b_0 \to b'\}$ independent of choice of path $b \rightsquigarrow b'$ inside U since U simply connected).

Seifert-Van Kampen theorem = given $X = U \cup V$, $U, V, U \cap V \subset X$ open & path connected this describes $\pi_1(X)$ in terms of $\pi_1(U)$ and $\pi_1(V)$. We've already seen a simpler statement: $\pi_1(X)$ is generated by the images of
$$\begin{tikzcd} \pi_1(U) \arrow[r, "i_*"] & \pi_1(X) \\ \pi_1(V) \arrow[ur, "j_*"'] & \end{tikzcd}.$$

* To formulate the thm, need to discuss the notion of free product of groups.

Assume G is a group, $G_1 \dots G_n$ subgroups of G which generate G, ie. any $x \in G$ can be written as $x = x_1 \dots x_m$ where each x_i is in some G_j. Also assume $G_j \cap G_k = \{1\}$ $\forall j \neq k$ $(x_1, \dots, x_m)$ is called a word of length m that represents x.

Say $(x_1 \dots x_m)$ is a reduced word if no G_j contains two consecutive elements x_i, x_{i+1}. (in particular if $m \geq 2$, no x_i can be $= 1$). (else can reduce to a shorter word $(x_1, \dots, x_i x_{i+1}, \dots, x_m)$)

Def: G is the free product of the subgroups $G_1 \dots G_n$, denoted $G = G_1 * \dots * G_n$, if G_i generate G, $G_i \cap G_j = \{1\}$, and every element of G is represented by a unique reduced word.

Ex: $\mathbb{Z}^2$ is not the free product of its two factors: denoting by a and b the two generators $(a = (1,0), b = (0,1))$, $ab = ba$ is represented by reduced words (a, b), (b, a), (a^2, b, a^{-1}), ...

* Alternative characterization: G is the free product of the subgroup G_j's iff, for any group H and any homomorphisms $h_j : G_j \to H$, $\exists$ unique homomorphism $h : G \to H$ s.t. (⭑)
$$\begin{tikzcd} G_j \arrow[r, hook] \arrow[rr, bend right, "h_j"'] & G \arrow[r, "h"] & H \end{tikzcd}$$
commutes $\forall j$.

(The point is: uniqueness of expression allows us to define $h(x_1 \dots x_m) = h_{j_1}(x_1) \dots h_{j_m}(x_m)$).
$\quad$ each $x_i \in G_{j_i}$

* External free product of groups G_j := group G + injective hom's $G_j \xrightarrow{i_j} G$ st. G is the free product of the subgroups $i_j(G_j)$.

Fact: This always exists! & unique up to iso.

Can be constructed as set of reduced words in G_j's (with product = concatenate + reduce). & satisfies universal property (⭑)

* In particular the free group on the elements $\{a_j\}$ is defined to be the external free product of cyclic groups $G_j = \{a_j^n \mid n \in \mathbb{Z}\} (\simeq \mathbb{Z})$

<u>Seifert-Van Kampen</u>: Let $X = U \cup V$, U and V open in X, $U \cap V$ path-connected $\ni x_0$.

The inclusions
$$\begin{tikzcd} & U \arrow[dr, hook, "j_1"] & \\ U\cap V \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & & X \\ & V \arrow[ur, hook, "j_2"'] & \end{tikzcd}$$
induce homomorphisms on π_1.

By univ. property of free product, $\exists$ unique homomorphism h st.
$$\begin{tikzcd} \pi_1(U,x_0) \arrow[d] \arrow[drr, bend left, "j_{1*}"] & & \\ \pi_1(U,x_0) * \pi_1(V,x_0) \arrow[rr, "h"] & & \pi_1(X,x_0) \\ \pi_1(V,x_0) \arrow[u] \arrow[urr, bend right, "j_{2*}"'] & & \end{tikzcd}$$
commutes.

(define h on words in elements of $\pi_1(U,x_0)$ and $\pi_1(V,x_0)$ using j_{1*} and j_{2*} on each component of the word!)

<u>Thm</u> (Seifert-Van Kampen):

this part is the generation result we saw last week

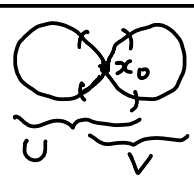
this is new.

The homomorphism h defined above is surjective, and its kernel N is the smallest normal subgroup of $\pi_1(U,x_0) * \pi_1(V,x_0)$ which contains all elements of the form $i_{1*}(g)^{-1}\, i_{2*}(g)$ $\forall g \in \pi_1(U\cap V, x_0)$. I.e. $\pi_1(X,x_0) \cong \pi_1(U,x_0) * \pi_1(V,x_0)/N$.

<u>Corollary 1</u> if $U \cap V$ is simply connected then $\pi_1(X,x_0) \cong \pi_1(U,x_0) * \pi_1(V,x_0)$.

<u>Corollary 2</u>: if V is simply connected then $\pi_1(X,x_0) \cong \pi_1(U,x_0)/N$, where N is the smallest normal subgroup containing the image of $i_{1*} : \pi_1(U\cap V, x_0) \to \pi_1(U,x_0)$.

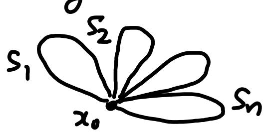
<u>Ex. 1</u>: figure 8:



$\Rightarrow$ U, V deformation retract onto circles
$U \cap V$ contractible

Hence $\pi_1(X,x_0) \cong \pi_1(U,x_0) * \pi_1(V,x_0) \cong \mathbb{Z} * \mathbb{Z}$ free group gen^d by loops around the two circles.

<u>Ex. 2</u>: by induction, wedge of n circles: $X = \bigcup_{i=1}^{n} S_i$, S_i homeo to S^1 $\forall i$, $S_i \cap S_j = \{x_0\}$.

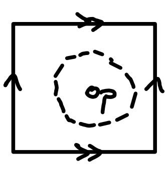


$\Rightarrow \pi_1(X,x_0) =$ free group on n generators $a_i =$ loops generating $\pi_1(S_i, x_0)$.

(Similarly for a finite graph with n loops).

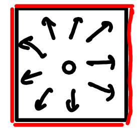
<u>Fundamental groups of surfaces</u> can also be calculated using Van Kampen!

eg. can now calculate π_1 of torus in a different way (easiest is still: $\mathbb{R}^2 \xrightarrow[p\times p]{\text{univ. cover}} T$).



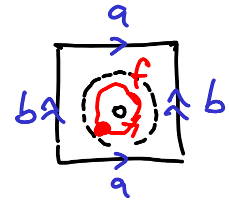
$T \simeq I \times I \big/ \begin{matrix} (x,0)\sim(x,1) & \forall x \\ (0,y)\sim(1,y) & \forall y. \end{matrix}$

Let $U = T \setminus \{p\}$
$V =$ open ball of radius $< \frac{1}{2}$ around p.

U deformation retracts onto wedge of two circles  $/\sim$

V is simply connected.

$U \cap V \simeq D^2 - pt$ has homotopy type of S^1.



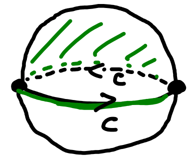
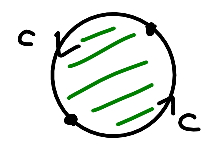
Using Corollary 2 above: $\pi_1(T) \cong \pi_1(U)/N$ where N is normal generated by the image of the loop f which generates $\pi_1(U \cap V)$ (and its conjugates)

$\pi_1(U)$ is a free group on gen's. a, b; and then the image of $[f]$ under the inclusion $U \cap V \hookrightarrow U$ is $aba^{-1}b^{-1}$

[the "obvious" picture  needs to be corrected slightly: base point should be fixed $\in U \cap V$!]

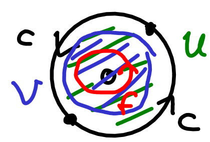
So we set $aba^{-1}b^{-1} = 1$ ie. $ab = (aba^{-1}b^{-1})ba = ba$, get abelian group $\simeq \mathbb{Z}^2$

$$\pi_1(T) \cong \underbrace{\langle a, b}_{\text{generators}} \,/\, \underbrace{ab = ba}_{\text{relations}} \rangle \cong \mathbb{Z} \times \mathbb{Z}.$$

- Similarly for $\pi_1(\mathbb{RP}^2)$, using $\mathbb{RP}^2 \simeq S^2/_{x \sim -x}$  $\simeq B^2/\sim$ 

  $x \sim -x \quad \forall x \in S^1 = \partial B^2$

  Now write $\mathbb{RP}^2 = U \cup V$, $\quad U = \mathbb{RP}^2 - \{p\}$
  $V =$ disc centred at p

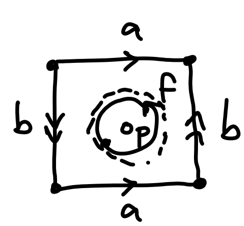
  U deformation retracts onto the boundary $S^1/_{x \sim -x} \underset{z \mapsto z^2}{\xrightarrow{\simeq}} S^1$
  so $\pi_1(U) \cong \mathbb{Z}$ w/ generator c.

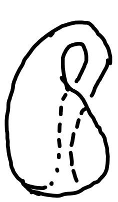
  V is simply connected. $\quad U \cap V \simeq D^2 - pt$ has homotopy type of S^1

  $\pi_1(\mathbb{RP}^2) \cong \pi_1(U)/N$, N normal subgroup generated by image of generator $[f] \in \pi_1(U \cap V)$ under inclusion, which is c^2.

  So $\pi_1(\mathbb{RP}^2) = \langle c \,/\, c^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

- <u>Klein bottle</u>: recall $K = I \times I/\sim$ $\quad (x,0) \sim (x,1)$
  $(0,y) \sim (1, 1-y)$

  Again write $K = U \cup V$, $\quad U = K - \{p\}$
  $V =$ disc centred at p $\quad \rightsquigarrow \pi_1(K) \cong \pi_1(U)/N$

  U retracts onto boundary $\cong$ figure 8 space  so
  $\pi_1(U) \cong$ free group on generators a, b.

$U \cap V$ has homotopy type of S^1, and the generator $[f] \in \pi_1(U \cap V) \cong \mathbb{Z}$
maps under inclusion to $aba^{-1}b$

So $\pi_1(K) \cong \langle a, b \mid aba^{-1}b = 1 \rangle$ not abelian: $ab = b^{-1}a$, not ba!

ie: $aba^{-1} = b^{-1}$: b conjugate to its inverse!

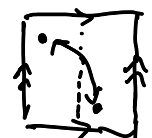
But this contains an index 2 subgroup H gen'd by a^2 and b, which commute! ($aba^{-1} = b^{-1}$ $\Rightarrow$ taking inverses, $ab^{-1}a^{-1} = b$, so

$$a^2ba^{-2} = a(aba^{-1})a^{-1} = ab^{-1}a^{-1} = b$$

So $a^2b = ba^2$ ✓). ($\Rightarrow$ subgroup $H \cong \mathbb{Z}^2$).

(can show, by rearranging letters via $ab = b^{-1}a$, this contains all words with even # of a's so it is an index 2 subgroup)

This subgroup corresponds to a deg. 2 covering map by the torus, $T \to K$!

T  $\to K$ quotient by $(x,y) \sim (x + \frac{1}{2}, 1-y)$

Ie. map $(x,y) \in I \times I/\sim_T$ to $\begin{cases} (2x, y) & \text{if } x \leq 1/2 \\ (2x-1, 1-y) & \text{if } x \geq \frac{1}{2} \end{cases}$ in $I \times I/\sim_K$.

Cool fact that this relates to: if you coat a klein bottle in paint all over, the paint forms a torus.