Math 55b Homework 7

Due Wednesday March 17, 2021.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.
- Questions marked * may be on the harder side.

Material covered: real analysis in one variable: sequences and series, differentiation, the Riemann integral, power series: Rudin chapters 3 (p.47-72), 5 (p.103-111), 6 (p.120-134), 7 (p.143-154), 8 (p.172-184), or Prof. McMullen's Math 55b notes sections 4-6 (ignore the Stieltjes integral).

0. Sometime over the weekend of March 13-14, please complete the week 7 feedback survey (in Canvas). This is important to help us fine-tune the course structure and pacing.

1. Suppose that $\sum a_n$ is a *divergent* series of real positive numbers $a_n > 0$, and denote by $s_n = a_1 + \cdots + a_n$ its partial sums.

(a) Show that $\displaystyle\sum \frac{a_n}{1+a_n}$ diverges.

(b) What can you say about $\displaystyle\sum \frac{a_n}{1+na_n}$ and $\displaystyle\sum \frac{a_n}{1+n^2a_n}$?

(c) Prove that $\displaystyle\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$, and deduce that $\displaystyle\sum \frac{a_n}{s_n}$ diverges.

(d) Prove that $\displaystyle\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$, and deduce that $\displaystyle\sum \frac{a_n}{s_n^2}$ converges.

(Hint for (a): it might be easier to treat separately the cases $a_n \to 0$ and $a_n \not\to 0$. Hint for (c): what does the divergence of $\sum a_n$ tell you about the sequence s_n?)

2. (a) For what values of x does the series $\displaystyle f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x^2}$ converge?

On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly?

(b) Is f continuous wherever the series converges? Is f bounded?

3. Let $p(t) = a_d t^d + a_{d-1}t^{d-1} + \cdots + a_0$ be a polynomial of degree d with *integer* coefficients, and suppose that $x \in \mathbb{R}$ satisfies $p(x) = 0$. Prove that there exists a constant $c > 0$ such that for any rational number $p/q \in \mathbb{Q}$ with $p/q \neq x$,

$$\left| x - \frac{p}{q} \right| \geq \frac{c}{q^d}.$$

Use this to show that $\alpha = \sum_0^\infty 10^{-n!}$ is transcendental (i.e. not the solution of any polynomial equation with integer coefficients).

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable, $|f''(x)| \leq 1$ for all $x \in \mathbb{R}$, and $\lim_{x\to+\infty} f(x) = 0$ (i.e., $\forall \epsilon > 0\ \exists M$ such that $x > M \Rightarrow |f(x)| < \epsilon$). Show that $\lim_{x\to+\infty} f'(x) = 0$.

(Hint: assuming $f'(x) = a > 0$, find a lower bound for f' over the interval $[x - a/2, x + a/2]$).

5. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\begin{cases} f(p/q) = 1/q & \text{if } p/q \in \mathbb{Q} \text{ is an irreducible fraction, } p, q \in \mathbb{Z},\ q > 0 \\ f(x) = 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

(a) At which points of $\mathbb{R}$ is f continuous?

(b) At which points of $\mathbb{R}$ is f differentiable?

(c) Show that f is the pointwise limit of a sequence of continuous functions.

(d) Show that f is Riemann integrable, and that $\int_0^1 f(x)\,dx = 0$.

(e)* (optional, extra credit) consider the function $g : \mathbb{R} \to \mathbb{R}$ defined similarly but with $g(p/q) = 1/q^3$ (and still $g(x) = 0$ for $x \notin \mathbb{Q}$). Find examples of points at which g is differentiable, and of points at which g is continuous but not differentiable. (Hint: use Problem 3.)

6. Let p, q be positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Show that if f, g are integrable, $f, g \geq 0$, and $\int_a^b f^p\,dx = \int_a^b g^q\,dx = 1$, then $\int_a^b fg\,dx \leq 1$. (Hint: first show that the inequality $uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q$ holds for all $u, v \in \mathbb{R}_{\geq 0}$; this can be done e.g. by studying the function $u \mapsto \frac{1}{p}u^p - uv$ for fixed v.)

(b) Use this to deduce *Hölder's inequality* for integrable functions:

$$\left| \int_a^b fg\,dx \right| \leq \left(\int_a^b |f|^p\,dx \right)^{1/p} \left(\int_a^b |g|^q\,dx \right)^{1/q}.$$

(c) (optional, extra credit) The L^p norm of $f \in C^0([a,b])$ is defined to be $\|f\|_p = \left(\int_a^b |f|^p\,dx \right)^{1/p}$. The triangle inequality for the L^p norm is known as *Minkowski's inequality*:

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Prove Minkowski's inequality for $p > 1$ by observing that $|f+g|^p \leq |f|\,|f+g|^{p-1} + |g|\,|f+g|^{p-1}$, and using Hölder's inequality to bound the integral of the right hand side.

7. Prove that the series given by the sum of the inverses of the primes, $\sum 1/p$, is divergent. (This shows that the primes are not too sparse as a subset of the positive integers).

Hint: given N, let $p_1, \ldots, p_k$ be the primes $\leq N$, and show that

$$\sum_{n=1}^{N} \frac{1}{n} \leq \prod_{j=1}^{k} \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \ldots \right) = \prod_{j=1}^{k} \left(1 - \frac{1}{p_j} \right)^{-1} \leq \exp\left(\sum_{j=1}^{k} \frac{2}{p_j} \right).$$

For the last inequality, show first that $(1-x)e^{2x} \geq 1$ for $x \in [0, \frac{1}{2}]$.

8. This problem gives a weaker form of Stirling's formula, which asserts that $n! \sim n^n e^{-n} \sqrt{2\pi n}$.

Let $\quad f(x) = (m + 1 - x)\log m + (x - m)\log(m+1) \quad$ for $x \in [m, m+1]$, $m \in \mathbb{Z}_{>0}$,

and $\quad g(x) = \frac{x}{m} - 1 + \log m \quad$ for $x \in [m - \frac{1}{2}, m + \frac{1}{2})$, $m \in \mathbb{Z}_{>0}$.

Describe or sketch the graphs of f and g, and show that $f(x) \leq \log x \leq g(x)$ for all $x \geq 1$. Then show that for n a positive integer,

$$\int_1^n f(x)\,dx = \log(n!) - \frac{1}{2}\log n \quad \text{and} \quad \int_1^n g(x)\,dx < \log(n!) - \frac{1}{2}\log n + \frac{1}{8}.$$

Conclude, by integrating $\log x$ over $[1, n]$, that

$$\frac{7}{8} < \log(n!) - (n + \frac{1}{2}) \log n + n < 1$$

for all integers $n \geq 2$, and hence that

$$e^{7/8}\, n^n e^{-n} \sqrt{n} < n! < e\, n^n e^{-n} \sqrt{n}.$$

9. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?