Iterated and Riemann integrals in several variables

* f continuous function on an n-cell $D = [a_1,b_1]\times\dots\times[a_n,b_n] \subset \mathbb{R}^n$

  $\Rightarrow$ we can define $\int_D f = \int_D f\, dx_1\dots dx_n = \int_D f\,|dx|$

  ↑ why? clearer after diff. forms

  either 1) as iterated integral:

  $$\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\dots\left(\int_{a_n}^{b_n} f(x_1\dots x_n)\,dx_n\right)\dots dx_2\right)dx_1 \qquad \text{or any order}$$

  2) as Riemann integral: split D into small cubes Q_i, and bound f between piecewise constant functions $s = s_i = \min f(Q_i)$ on $\mathrm{int}(Q_i)$, $S = S_i = \max f(Q_i)$ — " —

  $\rightarrow$ estimate $\sum s_i \,\mathrm{vol}(Q_i) \le \int_D f\,|dx| \le \sum S_i\, \mathrm{vol}(Q_i)$

  If f is continuous, hence uniformly continuous, then $\sup|S-s| \to 0$ as $\mathrm{diam}(Q_i)\to 0$, so this defines the integral uniquely.

<u>Fubini's thm</u> says: for continuous f, the iterated integrals for different orders of integration are all equal.

* if f is only piecewise continuous, integrability still holds if the regions of D where f is continuous are sufficiently regular - eg. delimited by smooth hypersurfaces.

  Specifically: when decomposing D into small cubes Q_i, want $\sum_{f_{|Q_i} \text{ not } C^0} \mathrm{vol}(Q_i) \longrightarrow 0$ as one subdivides further – over such cubes, $(S_i - s_i)$ doesn't $\to 0$ as step size $\to 0$, but if vol $\to 0$ we still have $\int_D (S-s)\,|dx| = \sum (S_i - s_i)\,\mathrm{vol}(Q_i) \to 0$.

* Thus we can define integrals over regions of $\mathbb{R}^n$ delimited by hypersurfaces by either
  - extending f by 0 outside of the given region, and integrating the resulting piecewise continuous function
  - using changes of coords. (via implicit function thm) to make the region of integration an n-cell. This requires change of variables...

<u>Thm:</u> $\varphi: U \to V$ diffeomorphism, f continuous on V, then

$$\int_V f(y)\,|dy| = \int_U f(\varphi(x))\,|\det D\varphi(x)|\, d|x|.$$

(won't prove. The geometric input is that if Q_i is a small cube $\ni x$ then $\varphi(Q_i) \approx$ small parallelepiped $\ni \varphi(x)$, with $\mathrm{vol}(\varphi(Q_i)) \sim |\det D\varphi(x)|\,.\,\mathrm{vol}(Q_i)$.)

* We also want to consider path integrals such as, given a path $\gamma \in C^1([0,1],\mathbb{R}^2)$, $\gamma(t)=(x(t),y(t))$
and a <u>differential</u> (1-form) $\omega = p(x,y)\,dx + q(x,y)\,dy$ (p,q C^0)
the path integral $\int_\gamma \omega = \int_\gamma p\,dx + q\,dy = \int_0^1 \big(p(\gamma(t))\,x'(t) + q(\gamma(t))\,y'(t)\big)\,dt$
  → this is independent of the parametrization of the path, by change of variable + chain rule.
  → if we reverse the path $(-\gamma)(t) = \gamma(1-t)$, then $\int_{-\gamma}\omega = -\int_\gamma \omega$.
  → given $f \in C^1(\mathbb{R}^2,\mathbb{R})$, define $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$, then $\int_\gamma df = f(\gamma(1)) - f(\gamma(0))$

This generalizes to arbitrary dimensions, using the language of <u>differential forms</u>.

* on $\mathbb{R}^n$, the symbols $dx_1,\dots,dx_n$ can be viewed as the differentials of the coordinate functions $x_1,\dots,x_n$; they form a basis of $T^* = \mathrm{Hom}(\mathbb{R}^n,\mathbb{R})$ linear forms on the space of tangent vectors $T \simeq \mathbb{R}^n$ ($dx_i(v) = v_i$ i^{th} component).
  Differential 1-forms are therefore functions with values in T^*.
* we now consider the <u>exterior powers</u> $\Lambda^k T^*$ = vector space with basis $\{dx_{i_1}\wedge\dots\wedge dx_{i_k} \mid i_1<\dots<i_k\}$, which are parts of the exterior algebra generated by T^*, ie. quotient of tensor algebra by setting $dx_i\wedge dx_j = -dx_j\wedge dx_i$. (NB: $\Lambda^0=\mathbb{R}$)
  ($\Rightarrow$ $\alpha\wedge\beta = -\beta\wedge\alpha$, $\alpha\wedge\alpha=0$ for all 1-forms).

<u>Def</u>: A <u>k-form</u> on an open subset $U\subset\mathbb{R}^n$ is a function with values in $\Lambda^k T^*$:
$$\omega = \sum_{i_1<\dots<i_k} p_{i_1\dots i_k}(x)\,dx_{i_1}\wedge\dots\wedge dx_{i_k}. \quad \text{(also denoted } = \sum_{|I|=k} p_I\,dx_I)$$

The space of C^∞ k-forms on $U\subset\mathbb{R}^n$ is usually denoted $\Omega^k(U)$ $(=C^\infty(U,\Lambda^k T^*))$
We can multiply k-forms by functions, or take exterior products ($\wedge: \Omega^k\times\Omega^\ell\to\Omega^{k+\ell}$)
$$(f\,dx_{i_1}\wedge\dots\wedge dx_{i_k})\wedge(g\,dx_{j_1}\wedge\dots\wedge dx_{j_\ell}) = (fg)\,dx_{i_1}\wedge\dots\wedge dx_{i_k}\wedge dx_{j_1}\wedge\dots\wedge dx_{j_\ell}$$
$$(=0 \text{ if } I\cap J\neq\emptyset, \quad = \pm(fg)\,dx_{I\cup J} \text{ if } I\cap J=\emptyset).$$

* The <u>exterior derivative</u> $d:\Omega^k\to\Omega^{k+1}$ is $d\left(\sum_I p_I\,dx_I\right) = \sum_{I,j}\frac{\partial p_I}{\partial x_j}dx_j\wedge dx_I$

<u>Eg</u>: $\Omega^0\to\Omega^1$: $df = \sum \frac{\partial f}{\partial x_i}dx_i$.
$\Omega^1(\mathbb{R}^2)\to\Omega^2(\mathbb{R}^2)$: $d(p\,dx+q\,dy) = \left(-\frac{\partial p}{\partial y}+\frac{\partial q}{\partial x}\right)dx\wedge dy$.

<u>Prop</u>: $d^2=0$ ie. $\forall\omega\in\Omega^k$, $d(d\omega)=0$.
(follows from: $\frac{\partial^2 p_I}{\partial x_j\partial x_k} = \frac{\partial^2 p_I}{\partial x_k\partial x_j}$, $dx_j\wedge dx_k + dx_k\wedge dx_j = 0$)

Say ω is <u>closed</u> if $d\omega = 0$, <u>exact</u> if $\omega = d\alpha$ for some $\alpha \in \Omega^{k-1}$.
The above says: exact $\Rightarrow$ closed.

<u>Thm</u> (Poincaré lemma): for $U \subset \mathbb{R}^n$ convex open, $1 \le k \le n$, $\omega \in \Omega^k$ is exact iff ω is closed.

<u>Remark:</u> This leads to <u>de Rham cohomology</u>, a key invariant in diff. topology!
$$H^k_{dR}(U) := \ker(d : \Omega^k(U) \to \Omega^{k+1}(U)) / \mathrm{Im}(d : \Omega^{k-1}(U) \to \Omega^k(U)) = \{\text{closed } k\text{-forms}\}/\{\text{exact}\}.$$
The Poincaré lemma says $H^k_{dR}(U) = 0$ for $U \subset \mathbb{R}^n$ convex and $k \ge 1$
while $H^1_{dR}(\mathbb{R}^2 - \{0\}) \ne 0$ detects $\mathbb{R}^2 - \{0\}$ isn't simply connected.

* <u>Pullback of differential forms:</u> if $\varphi : U \to V$ is a smooth map ($U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$) then we have a map $\varphi^* : \Omega^k(V) \to \Omega^k(U)$ characterized by
$$\begin{cases} (1) \text{ for functions } (k=0),\ \varphi^*(f) = f \circ \varphi \\ (2)\ \varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta \\ (3)\ \varphi^*(d\alpha) = d(\varphi^*\alpha). \end{cases}$$
Concretely, denoting by (x_i) coords. on U, (y_j) on V, $\varphi^*(dy_j) = d(y_j \circ \varphi) = \sum_i \frac{\partial \varphi_j}{\partial x_i} dx_i$ $(= d\varphi_j)$
and
$$\varphi^*\Big(\sum_J p_J(y)\, dy_{j_1} \wedge \dots \wedge dy_{j_k}\Big) = \sum_J p_J(\varphi(x))\, \underbrace{d\varphi_{j_1} \wedge \dots \wedge d\varphi_{j_k}}_{= \sum_I \det\left(\frac{\partial(\varphi_{j_1}, \dots, \varphi_{j_k})}{\partial(x_{i_1}, \dots, x_{i_k})}\right) dx_I}$$
Especially: for $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ and $k = n$,
$$\varphi^*(dy_1 \wedge \dots \wedge dy_n) = (\det D\varphi)\, dx_1 \wedge \dots \wedge dx_n$$

* <u>Integration of differential forms:</u>
given $\omega = \sum_I p_I(x)\, dx_I \in \Omega^k(U)$, we can integrate ω over a k-dimensional submanifold $M \subset U$ parametrized by a smooth map from a k-cell $D \subset \mathbb{R}^k$ to $U \subset \mathbb{R}^n$ (or any other nice enough domain for integration), $\varphi : D \hookrightarrow U$, $t \mapsto (\varphi_1(t), \dots, \varphi_n(t))$, $M = \varphi(D)$,
by setting
$$\boxed{\int_M \omega = \int_D \sum_I p_I(\varphi(t)) \det\left(\left(\frac{\partial \varphi_i}{\partial t_j}\right)_{\substack{i \in I \\ 1 \le j \le k}}\right) |dt|.}$$

<u>check:</u> for 1-forms this agrees with path integral formula $\int_\gamma p_i\, dx_i = \int p_i(\gamma(t)) \frac{dx_i}{dt}\, dt$

What this formula means is:
$$\begin{cases} \bullet \text{ for } n\text{-forms on } D \subset U \subset \mathbb{R}^n,\quad \int_D f\, dx_1 \wedge \dots \wedge dx_n = \int_D f\, |dx|. \\ \bullet \text{ for general } \varphi : D^k \to U \subset \mathbb{R}^n,\quad \int_{\varphi(D)} \omega = \int_D \varphi^* \omega \end{cases}$$
$\leftarrow$ k-form on $D \subset \mathbb{R}^k$ $\Rightarrow$ usual integral.

* Can similarly integrate k-forms over $M =$ finite union of parametrized pieces.

* **pullback formula** given a smooth map $\varphi : U \subset \mathbb{R}^m \to V \subset \mathbb{R}^n$, $\omega \in \Omega^k(V)$, and $M^k \subset U$:
$$\int_{\varphi(M)} \omega = \int_M \varphi^* \omega .$$

This is basically equivalent to change of variables formula for usual $\int_D f\, |dx|$, and implies that $\int_M \omega$ is independent of the manner in which we parametrize M as the image of a map $\varphi : D \to U$ (or union of pieces) as long as all reparametrizations are orientation-preserving (ie. we compose $\varphi : D \to U$ with a diffeomorphism $g : D' \to D$ st. $\det(Dg) > 0$ everywhere). ($D' \subset \mathbb{R}^k$, $D \subset \mathbb{R}^k$)

Ex: $\omega = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$ on $\mathbb{R}^2 - \{0\}$, $C_r =$ circle of radius r, oriented counterclockwise: (as path $(r,0) \to (r,0)$)

Pulling back via $\varphi : (r,\theta) \mapsto (r\cos\theta, r\sin\theta)$, (polar coordinates),
$$\varphi^*\omega = \frac{(r\cos\theta)(r\cos\theta\, d\theta) - (r\sin\theta)(-r\sin\theta\, d\theta)}{r^2} = d\theta$$

So $\displaystyle\int_{C_r} \omega = \int_{\{r\}\times[0,2\pi]} \varphi^*\omega = \int_0^{2\pi} d\theta = 2\pi$ (independent of r)

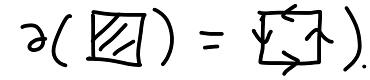
Note: $d\omega = 0$ (by direct calc. or using $\varphi^*(d\omega) = d(\varphi^*\omega) = d(d\theta) = 0$)

ie. ω is closed; but not exact! if $\exists f(x,y)$ on $\mathbb{R}^2 - \{0\}$ st $df = \omega$ then path integral $\int_{C_r} \omega = \int_{C_r} df = f(r,0) - f(r,0) = 0$. $\quad H^1_{dR}(\mathbb{R}^2 - 0) \neq 0$.

But... path integral is independent of radius r, or in fact same for any

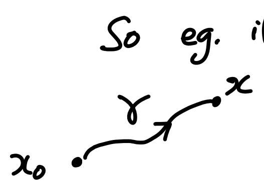
This is a consequence of **Stokes' theorem**.

for $M \subset \mathbb{R}^n$ parametrized as $\varphi(D)$, $D \subset \mathbb{R}^k$ k-cell (or other nice domain) define $\partial M = (k-1)$-dimensional boundary $\varphi(\partial D)$ (for $D = \prod [a_i, b_i]$ a k-cell, this consists of $2k$ pieces...), with suitable orientation.

(most relevant to us: )

Stokes' thm: $\forall \omega \in \Omega^{k-1}$, $\displaystyle\int_M d\omega = \int_{\partial M} \omega$.

So eg. if ω is a closed 1-form on a simply connected $U \subset \mathbb{R}^n$, the path-integral $\int_\gamma \omega$ is indept of choice of path γ from base point x_0 to x.



Indeed, path-independence comes from Stokes for the surface S traced by a path homotopy:

$$d\omega = 0 \Rightarrow \quad 0 = \int_S d\omega = \int_{\partial S = \gamma - \gamma'} \omega = \int_\gamma \omega - \int_{\gamma'} \omega$$

So we can define $f(x) = \int_\gamma \omega$ for any path $\gamma : x_0 \to x$.

Stokes again (= fund. thm. calc.) gives $\int_\gamma df = f(x) - f(x_0) = \int_\gamma \omega \quad \forall$ path γ,

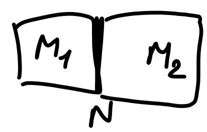
and we find that $\omega = df$ is exact. ($\Rightarrow$ Poincaré lemma).

<u>Rmk:</u> Stokes' theorem for diff. forms in $\mathbb{R}^2$ and $\mathbb{R}^3$ specializes to all the theorems of multivariable calculus
$$\left(\begin{array}{l} k=0 : \text{fund. thm. of calc. for path integrals} \\ k=1 : \text{Green's theorem in } \mathbb{R}^2, \text{ curl in } \mathbb{R}^3 \\ k=2 \text{ in } \mathbb{R}^3 : \text{Gauss / divergence thm.} \end{array}\right)$$

The most useful case for cx. analysis is: $D \subset \mathbb{R}^2$ 

$\Rightarrow \int_{\partial D} p\,dx + q\,dy = \int_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx \wedge dy.$

<u>Sketch proof:</u>

- both sides obey pullback formula (using $\varphi^* d\omega = d(\varphi^* \omega)$, and $\partial \varphi(M) = \varphi(\partial M)$).
  so can do changes of coordinates / pullback by parametrization $D \xrightarrow[\varphi]{} M$.
- can decompose into pieces (either by writing ω as sum of forms with support contained in subsets that have a single parametrization, or by observing that if $M = M_1 \cup M_2$, $M_1 \cap M_2 = N \subset \partial M_i$ 

then ∂M_1 and ∂M_2 contain N with opposite orientations, and so
$$\int_M d\omega = \int_{M_1} d\omega + \int_{M_2} d\omega \quad \& \quad \int_{\partial M} \omega = \int_{\partial M_1} \omega + \int_{\partial M_2} \omega.$$
- over a k-cell, and considering each component of $\omega \in \Omega^{k-1}$ separately: eg.
  $D = \prod_{i=1}^{k} [a_i, b_i] = D' \times [a_k, b_k]$: $\quad \omega = f\, dx_1 \wedge \dots \wedge dx_{k-1} \Rightarrow d\omega = (-1)^{k-1} \frac{\partial f}{\partial x_k} dx_1 \wedge \dots \wedge dx_{k-1} \wedge dx_k$

$$\int_D d\omega = \int_D (-1)^{k-1} \frac{\partial f}{\partial x_k} |dx| \overset{\substack{\text{iterated} \\ \text{integral}}}{=} \int_{D'} \left(\int_{a_k}^{b_k} (-1)^{k-1} \frac{\partial f}{\partial x_k} dx_k \right) dx_1 \dots dx_{k-1}$$

$$\underset{\text{fund. th. calc.}}{=} (-1)^{k-1} \int_{D'} \left(f(x_1, \dots x_{k-1}, b_k) - f(x_1, \dots x_{k-1}, a_k) \right) dx_1 \dots dx_{k-1}$$

$$= (-1)^{k-1} \left(\int_{D' \times \{b_k\}} \omega - \int_{D' \times \{a_k\}} \omega \right) = \int_{\partial D} \omega$$

using that $\int \omega$ vanishes on the other faces of D ($\perp (x_1, \dots x_{k-1})$-plane) and orientation convention for ∂D (which we didn't state but is designed to make this work).